ON THE EXISTENCE OF TWISTED SHALIKA PERIODS: THE ARCHIMEDEAN CASE

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ABSTRACT. Let $\mathbb{K}$ be an archimedean local field. We investigate the existence of the twisted Shalika functionals on irreducible admissible smooth representations of $\mathrm{GL}_{2n}(\mathbb{K})$ in terms of their L-parameters. As part of our proof, we establish a Hochschild-Serre spectral sequence for nilpotent normal subgroups and a Künneth formula in the framework of Schwartz homology. We also prove the analogous result for twisted linear periods using theta correspondence. The existence of twisted Shalika functionals on representations of $\mathrm{GL}_{2n}^{+}(\mathbb{R})$ is also studied, which is of independent interest.

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1. Introduction

1.1. **Main results.** Let $\mathbb{K}$ be a local field of characteristic zero. The Shalika subgroup of the general linear group $\mathrm{GL}_{2n}(\mathbb{K})$ ($n \geq 1$) is defined to be

$$S_{2n}(\mathbb{K}) := \left\{ \begin{bmatrix} g & 0 \\ 0 & g \end{bmatrix} \cdot \begin{bmatrix} 1_n & b \\ 0 & 1_n \end{bmatrix} \ : \ g \in \mathrm{GL}_n(\mathbb{K}),\ b \in \mathrm{M}_n(\mathbb{K}) \right\},$$

where M_n indicates the algebra of $n \times n$ matrices. Let η be a character of $\mathbb{K}^\times$, and let ψ be a non-trivial unitary character of $\mathbb{K}$. Define a character $\xi_{\eta,\psi}$ on $S_{2n}(\mathbb{K})$ by

$$\xi_{\eta,\psi}\left(\begin{bmatrix} g & 0 \\ 0 & g \end{bmatrix} \cdot \begin{bmatrix} 1_n & b \\ 0 & 1_n \end{bmatrix} \right) := \eta(\det(g))\psi(\mathrm{Tr}(b)).$$

For an admissible smooth representation π of $\mathrm{GL}_{2n}(\mathbb{K})$, we say that π has a non-zero (η, ψ)**-twisted Shalika period** if

$$\mathrm{Hom}_{S_{2n}(\mathbb{K})}(\pi, \xi_{\eta,\psi}) \neq \{0\}.$$

When η is the trivial character, we shall refer to it simply as the **Shalika period**. Here and henceforth, for the archimedean local field $\mathbb{K}$, by an admissible smooth representation of $\mathrm{GL}_n(\mathbb{K})$, we mean a Casselman-Wallach representation of it. Recall that a representation of a real reductive group is called a Casselman-Wallach representation if it is Fréchet, smooth, admissible, and of moderate growth. The readers may consult [Wal92, Chapter 11] or [BK14] for details. It is proved in [CS20] that the space $\mathrm{Hom}_{S_{2n}(\mathbb{K})}(\pi, \xi_{\eta,\psi})$ is at most one-dimensional if π is irreducible.

A natural question is when the space $\mathrm{Hom}_{S_{2n}(\mathbb{K})}(\pi, \xi_{\eta,\psi})$ is non-zero, which is commonly referred to as the distinction problem. In the framework of the relative Langlands program, this question is related to the BZSV quadruple

$$(\mathrm{GL}_{2n} \times \mathrm{GL}_1, \mathrm{GL}_n, 0, \iota : \mathrm{GL}_n \times \mathrm{SL}_2 \to \mathrm{GL}_{2n} \times \mathrm{GL}_1),$$

where

$$\begin{array}{cccc} \iota : & \mathrm{GL}_n \times \mathrm{SL}_2 & \to & \mathrm{GL}_{2n} \times \mathrm{GL}_1 \\ & (A, B) & \mapsto & (A \otimes B, \det A), \end{array}$$

while the dual BZSV quadruple of it is

$$(\mathrm{GL}_{2n} \times \mathrm{GL}_1, \mathrm{GSp}_{2n}, 0, (i, \lambda) : \mathrm{GSp}_{2n} \to \mathrm{GL}_{2n} \times \mathrm{GL}_1),$$

where $i : \mathrm{GSp}_{2n} \to \mathrm{GL}_{2n}$ is the inclusion, $\lambda : \mathrm{GSp}_{2n} \to \mathrm{GL}_1$ is the similitude character of GSp_{2n}. The local aspect of this framework suggests that the dual quadruple is related to the distinction problem, which will be confirmed by our main result.

From now on, we will assume that $\mathbb{K}$ is an archimedean local field. Let $W_{\mathbb{K}}$ be the Weil group of $\mathbb{K}$.

Definition 1.1. *For an L-parameter $\phi : W_{\mathbb{K}} \to \mathrm{GL}_{2n}(\mathbb{C})$, its **$\eta$-extension** is defined to be*

$$\begin{array}{cccccc} \phi^{(\eta)} : & W_{\mathbb{K}} & \to & \mathrm{GL}_{2n}(\mathbb{C}) & \times & \mathrm{GL}_1(\mathbb{C}) \\ & x & \mapsto & (\ \phi(x) & , & \eta(r(x))) \end{array} .$$

where $r : W_{\mathbb{K}} \to \mathbb{K}^\times$ is the reciprocity map (will be recalled in Subsection 2.1).

*An L-parameter ϕ is said to be **of η-symplectic type**, if its η-extension $\phi^{(\eta)}$ factors through the above map $(i, \lambda) : \mathrm{GSp}_{2n}(\mathbb{C}) \to \mathrm{GL}_{2n}(\mathbb{C}) \times \mathrm{GL}_1(\mathbb{C})$.*

The main result of this paper is the following.

Theorem 1.2. *Let π be an irreducible Casselman-Wallach representation of $\mathrm{GL}_{2n}(\mathbb{K})$.*

(A) (§5.3) *If π has a non-zero (η,ψ)-twisted Shalika period, then its L-parameter is of η-symplectic type.*
(B) (§6) *Assume that π is generic. If the L-parameter of π is of η-symplectic type, then π admits a (η,ψ)-twisted Shalika period.*

To summarize, we have the following equivalence between the existence of the twisted Shalika period and the type of L-parameter.

Theorem 1.3. *For an irreducible generic Casselman-Wallach representation π of $\mathrm{GL}_{2n}(\mathbb{K})$, it has a non-zero (η,ψ)-twisted Shalika period if and only if its L-parameter is of η-symplectic type.*

Remark. The generic condition cannot be removed without additional assumptions. For example, the trivial representation of $\mathrm{GL}_2(\mathbb{R})$ is not a generic representation. It doesn't have Shalika periods, while its L-parameter is of symplectic type.

When π is an irreducible essentially tempered cohomological representation, Theorem 1.3 has been proved in [CJLT20] for $\mathbb{K}=\mathbb{R}$ and [LT20] for $\mathbb{K}=\mathbb{C}$. When π is a principal series representation, [JLST24] also explores the existence of the twisted Shalika periods under certain conditions. When η is a trivial character, Theorem 1.3 is proved in [Mat17] for the non-archimedean case and [ALMSY24] for the archimedean case.

All the aforementioned works (except [CJLT20] and [JLST24]) study the analogous properties of twisted linear periods, then transfer them to the twisted Shalika periods using the analytic method. By comparison, Our proof provides a direct analysis of the twisted Shalika periods and has no restriction on η.

The theory of theta correspondence gives a way to relate different kinds of periods. Let $H_{2n}(\mathbb{K}) := \mathrm{GL}_n(\mathbb{K}) \times \mathrm{GL}_n(\mathbb{K})$, and view it as a subgroup of $\mathrm{GL}_{2n}(\mathbb{K})$ via the diagonal embedding. For an admissible smooth representation π of $\mathrm{GL}_{2n}(\mathbb{K})$, we say that π has a non-zero **η-twisted linear period** if

$$\mathrm{Hom}_{H_{2n}(\mathbb{K})}(\pi, (\eta\circ\det)\boxtimes\mathbb{C}) \neq \{0\}.$$

Using theta correspondence, it is shown in [Gan19] that Shalika periods are related to linear periods over non-archimedean local fields. The following theorem is an Archimedean analog of it.

Theorem 1.4 (Theorem 7.5)**.** *Let $\mathbb{K}$ be an archimedean local field. Let π be an irreducible Casselman-Wallach representation of $\mathrm{GL}_{2n}(\mathbb{K})$ and χ be a character of $\mathrm{GL}_n(\mathbb{K})$. Denote by $\Theta(\pi)$ the big theta lift of a representation π, one has*

$$\mathrm{Hom}_{\mathrm{GL}_n^{\Delta}(\mathbb{K})\ltimes U}(\Theta(\pi),\chi\boxtimes\psi) \cong \mathrm{Hom}_{\mathrm{GL}_n(\mathbb{K})\times\mathrm{GL}_n(\mathbb{K})}(\pi^{\vee},\chi\boxtimes\mathbb{C}),$$

where $S_{2n}(\mathbb{K}) = \mathrm{GL}_n^{\Delta}(\mathbb{K}) \ltimes U$. Moreover, if π is a generic representation, we have

$$\mathrm{Hom}_{\mathrm{GL}_n^{\Delta}(\mathbb{K})\ltimes U}(\pi,\chi\boxtimes\psi) \cong \mathrm{Hom}_{\mathrm{GL}_n(\mathbb{K})\times\mathrm{GL}_n(\mathbb{K})}(\pi,\chi\boxtimes\mathbb{C})$$

Combining Theorem 1.3 and Theorem 1.4, we get the following corollary.

Corollary 1.5. *Let $\mathbb{K}$ be an archimedean local field. Let π be an irreducible generic Casselman-Wallach representation of $\mathrm{GL}_{2n}(\mathbb{K})$. The following are equivalent:*
(1) The L-parameter of π is of η-symplectic type.
(2) π has a (η,ψ)-twisted Shalika periods.
(3) π has a η-twisted linear periods.

Twisted Shalika periods are important in the study of L-functions of symplectic type. In [JST24], they study the period relations for the critical values of the standard L-functions for an irreducible regular algebraic cuspidal automorphic representations of

$\mathrm{GL}_{2n}(\mathbb{A})$ of symplectic type. During their proof, a constant ϵ_π has been defined (and will be recalled in Section 8, (8.1)), which reflects the existence of the twisted Shalika periods on the representations of $\mathrm{GL}_{2n}^+(\mathbb{R})$. The following theorem provides the calculation of ϵ_π.

Theorem 1.6 (Theorem 8.6). *The notation is as shown in Section 8. Let π be an irreducible generic Casselman-Wallach representation of $\mathrm{GL}_{2n}(\mathbb{R})$. For $a \in \mathbb{R}^\times$, denote*

$$\psi_a : \mathbb{R} \to \mathbb{C}^\times, x \mapsto \exp(2\pi a x\sqrt{-1}).$$

Assume π admits a non-zero (η, ψ_a)-twisted Shalika period and $\pi|_{\mathrm{GL}_{2n}^+(\mathbb{R})}$ is reducible, then π has form

$$D_{k_1,\lambda_1} \dot{\times} \cdots \dot{\times} D_{k_n,\lambda_n}, \;\; k_i \in \mathbb{Z}_{\geq 0}, \;\; \lambda_i \in \mathbb{C}.$$

Let $p := \#\{1 \leq i \leq n | D_{k_i,\lambda_i}$ has (η, ψ_a)-twisted Shalika period$\}$ and $q := \frac{n-p}{2}$, which are both integers since ϕ_π is of η-symplectic type. Then

$$\epsilon_\pi = (\mathrm{sgn}\;\; a)^p \cdot (-1)^{\frac{p(p-1)}{2}+q}.$$

1.2. **Outline of the proof.** We outline a general framework for addressing such problems as Theorem 1.2A and Theorem 1.2B, and specialize it to our case at the end. We shall use the terminology in Schwartz homology developed in [CS21], which will be reviewed in Section 3. The proof of Theorem 1.2A is inspired by [ST23].

Let G be an almost linear Nash group and let H be a Nash subgroup of G. Denote by $\mathcal{S}\mathrm{mod}_G$ the category of smooth Fréchet representations of G of moderate growth. Let X be a G-Nash manifold and let $\mathcal{E}$ be a tempered G-vector bundle on X. Denote by $\Gamma^\varsigma(X, \mathcal{E})$ the Schwartz sections of the tempered bundle $\mathcal{E}$ over X, then $\Gamma^\varsigma(X, \mathcal{E}) \in \mathcal{S}\mathrm{mod}_G$. Take $x \in X$, denote by $\mathcal{E}_x$ the fiber of $\mathcal{E}$ at x and H_x the stabilizer in H at x. Let χ be a character of H. Note that $\mathrm{Hom}_H(\Gamma^\varsigma(X, \mathcal{E}), \chi)$ is the continuous linear dual of the zeroth Schwartz homology $\mathrm{H}_0^{\mathcal{S}}(H, \Gamma^\varsigma(X, \mathcal{E}) \otimes \chi^{-1})$. In the following, we investigate the relationship between $\mathrm{H}_*^{\mathcal{S}}(H, \Gamma^\varsigma(X, \mathcal{E}) \otimes \chi^{-1})$ and $\mathrm{H}_*^{\mathcal{S}}(H_x, \mathcal{E}_x \otimes \chi^{-1})$.

Assume that X admits a finite decreasing sequence of open submanifolds

$$U_1 := X \supsetneq U_2 \supsetneq ... \supsetneq U_r \supsetneq U_{r+1} := \varnothing,$$

such that for $i \in \{1, \ldots, r\}$, $O_i := U_i \backslash U_{i+1}$ is an H-orbit in X. For each $i \in \{1, \ldots, r\}$, we have the following short exact sequence

$$0 \to \Gamma^\varsigma(U_{i+1}, \mathcal{E}) \to \Gamma^\varsigma(U_i, \mathcal{E}) \to \Gamma^\varsigma_{O_i}(U_i, \mathcal{E}) \to 0, \tag{1.1}$$

where $\Gamma^\varsigma_{O_i}(U_i, \mathcal{E}) := \Gamma^\varsigma(U_i, \mathcal{E})/\Gamma^\varsigma(U_{i+1}, \mathcal{E})$. After twisting the character χ^{-1} of H and taking Schwartz homology with respect to H, we get the long exact sequence

$$\begin{aligned}\cdots \to \mathrm{H}_j^{\mathcal{S}}(H, \Gamma^\varsigma(U_{i+1}, \mathcal{E}) \otimes \chi^{-1}) \to \mathrm{H}_j^{\mathcal{S}}(H, \Gamma^\varsigma(U_i, \mathcal{E}) \otimes \chi^{-1}) \to \\ \mathrm{H}_j^{\mathcal{S}}(H, \Gamma^\varsigma_{O_i}(U_i, \mathcal{E}) \otimes \chi^{-1}) \to \cdots .\end{aligned} \tag{1.2}$$

We may encounter two different situations in practice. In the first case (corresponding to the case of Theorem 1.2A), we know that $\mathrm{H}_0^{\mathcal{S}}(H, \Gamma^\varsigma(X, \mathcal{E}) \otimes \chi^{-1}) \neq 0$, and we want to deduce expected properties on the representation $\mathcal{E}_x$ of H_x. Following from the long exact sequence (1.2), we have

$$\dim \mathrm{H}_j^{\mathcal{S}}(H, \Gamma^\varsigma(X, \mathcal{E}) \otimes \chi^{-1}) \leq \sum_{i=1}^{r} \dim \mathrm{H}_j^{\mathcal{S}}(H, \Gamma^\varsigma_{O_i}(U_i, \mathcal{E}) \otimes \chi^{-1}).$$

Thus we know there exist an $l \in \{1, \ldots, r\}$ such that

$$\dim \mathrm{H}_0^{\mathcal{S}}(H, \Gamma^\varsigma_{O_l}(U_l, \mathcal{E}) \otimes \chi^{-1}) \neq 0.$$

In the second case (corresponding to the case of Theorem 1.2B), we know that $\mathrm{H}_0^{\mathcal{S}}(H, \Gamma^{\varsigma}(\mathcal{O}_r, \mathcal{E}) \otimes \chi^{-1}) \neq 0$ for the open orbit $\mathcal{O}_r \subset X$, and we want to prove that the map induced by extension by zero is an isomorphism on the zeroth homology, i.e.

$$\mathrm{H}_0^{\mathcal{S}}(H, \Gamma^{\varsigma}(\mathcal{O}_r, \mathcal{E}) \otimes \chi^{-1}) \cong \mathrm{H}_0^{\mathcal{S}}(H, \Gamma^{\varsigma}(X, \mathcal{E}) \otimes \chi^{-1}).$$

Then it suffices to prove that for $i \in \{1, \ldots, r-1\}$,

$$\mathrm{H}_0^{\mathcal{S}}(H, \Gamma^{\varsigma}_{O_i}(U_i, \mathcal{E}) \otimes \chi^{-1}) = \mathrm{H}_1^{\mathcal{S}}(H, \Gamma^{\varsigma}_{O_i}(U_i, \mathcal{E}) \otimes \chi^{-1}) = 0.$$

To summarize, in both cases, we reduce the problem to the homological property of $\Gamma^{\varsigma}_{O_i}(U_i, \mathcal{E})$.

Using Borel's lemma (Proposition 3.1), $\Gamma^{\varsigma}_{O_i}(U_i, \mathcal{E})$ admits a decreasing filtration $\Gamma^{\mathcal{S}}_{O_i}(U_i, \mathcal{E})_k,\ k = 0, 1, 2, \ldots$, such that

$$\Gamma^{\mathcal{S}}_{O_i}(U_i, \mathcal{E}) \cong \varprojlim_k \Gamma^{\mathcal{S}}_{O_i}(U_i, \mathcal{E}) / \Gamma^{\mathcal{S}}_{O_i}(U_i, \mathcal{E})_k,$$

with the graded pieces

$$\Gamma^{\mathcal{S}}_{O_i}(U_i, \mathcal{E})_k / \Gamma^{\mathcal{S}}_{O_i}(U_i, \mathcal{E})_{k+1} \cong \Gamma^{\mathcal{S}}(O_i, \mathrm{Sym}^k(\mathcal{N}^*_{O_i}(U_i)) \otimes \mathcal{E}), \quad (k \geq 0).$$

Write $\Gamma^{\mathcal{S}}_{i,k} := \Gamma^{\mathcal{S}}_{O_i}(U_i, \mathcal{E})_k,\ k = 0, 1, 2, \ldots$ for short. We have a short exact sequence

$$0 \to \Gamma^{\mathcal{S}}_{i,k} / \Gamma^{\mathcal{S}}_{i,k+1} \to \Gamma^{\mathcal{S}}_{i,0} / \Gamma^{\mathcal{S}}_{i,k+1} \to \Gamma^{\mathcal{S}}_{i,0} / \Gamma^{\mathcal{S}}_{i,k} \to 0,$$

and hence a long exact sequence

$$(1.3) \quad \cdots \to \mathrm{H}_j^{\mathcal{S}}(H, (\Gamma^{\mathcal{S}}_{i,k} / \Gamma^{\mathcal{S}}_{i,k+1}) \otimes \chi^{-1}) \to \mathrm{H}_j^{\mathcal{S}}(H, (\Gamma^{\mathcal{S}}_{i,0} / \Gamma^{\mathcal{S}}_{i,k+1}) \otimes \chi^{-1}) \to \mathrm{H}_j^{\mathcal{S}}(H, (\Gamma^{\mathcal{S}}_{i,0} / \Gamma^{\mathcal{S}}_{i,k}) \otimes \chi^{-1}) \to \cdots.$$

Assume that

$$(1.4) \qquad \dim \mathrm{H}_{j+1}^{\mathcal{S}}(H, (\Gamma^{\mathcal{S}}_{i,0} / \Gamma^{\mathcal{S}}_{i,k}) \otimes \chi^{-1}) < \infty, \quad \forall\, k \geq 0.$$

According to Proposition 3.5, we have

$$\mathrm{H}_j^{\mathcal{S}}(H, \Gamma^{\mathcal{S}}_{\mathcal{O}_i}(U_i, \mathcal{E}) \otimes \chi) \cong \varprojlim_k \mathrm{H}_j^{\mathcal{S}}(H, (\Gamma^{\mathcal{S}}_{i,0} / \Gamma^{\mathcal{S}}_{i,k}) \otimes \chi).$$

Using Borel's lemma and the long exact sequence (1.3), we may reduce the problem to the calculation of

$$\mathrm{H}_j^{\mathcal{S}}(H, \Gamma^{\mathcal{S}}(O_i, \mathrm{Sym}^k(\mathcal{N}^*_{O_i}(U_i)) \otimes \mathcal{E}) \otimes \chi^{-1}),$$

which, using Shapiro's lemma (Propositon 3.2), is related to the homological property of $\mathcal{E}_x$ with respect to H_x for $x \in \mathcal{O}_i$.

Now we return to our case. Let $X := P \backslash \mathrm{GL}_{2n}(\mathbb{K})$ be a partial flag manifold, $\mathcal{E}$ the tempered vector bundle on X associated with a standard module, $S := S_{2n}(\mathbb{K})$ the Shalika subgroup, and $\chi := \xi_{\eta,\psi}$.

To prove Theorem 1.2A, we first describe the S-orbit decomposition of X in Section 4, and give the definition of matching orbits. Then we prove in Lemma 5.5 that the homology groups occurring in (1.4) are finite-dimensional. Finally, using various tools of the Schwartz homology and the calculation of the Jacquet module, we prove that

$$\mathrm{H}_0^{\mathcal{S}}(S, \Gamma^{\varsigma}_{O_i}(U_i, \mathcal{E}) \otimes \xi^{-1}_{\eta,\psi}) \cong \begin{cases} \mathrm{H}_0^{\mathcal{S}}(S, \Gamma^{\varsigma}(\mathcal{O}_i, \mathcal{E}) \otimes \xi^{-1}_{\eta,\psi}), & \text{if } \mathcal{O}_i \text{ is a matching orbit;} \\ 0, & \text{otherwise.} \end{cases}$$

and deduce Theorem 1.2A from this using Proposition 5.12.

The existence of the twisted Shalika functional on the parabolic induced representation has been studied in [CJLT20, Theorem 2.1]. Using this, we reduce the proof of Theorem

1.2B to the case for $\mathrm{GL}_4(\mathbb{R})$. Using the framework described above, we prove that, in this case, for all non-open orbits $\mathcal{O}_i$ and $\forall j \in \mathbb{Z}$,

$$\mathrm{H}_j^{\mathcal{S}}(S_4(\mathbb{R}), \Gamma^{\varsigma}_{O_i}(U_i, \mathcal{E}) \otimes \xi^{-1}_{\eta,\psi}) = 0,$$

and then deduce the result from the long exact sequence (1.2).

The proof of Theorem 1.4 is similar to the non-archimedean case as in [Gan19]. However, the calculation of the coinvariance space of the Weil representation is more involved in the archimedean case. We use [AGS15, Lemma 6.2.2] to show that the result is just the same as in the non-archimedean case.

For proving Theorem 1.6, we study the restriction of certain parabolic induced representations to $\mathrm{GL}^+_{2n}(\mathbb{R})$, then the similar arguments as in Section 6 can be applied.

1.3. **Structure of the paper.** We now describe the contents and the organization of this paper. In Section 2, we give the necessary preliminaries on the local Langlands correspondence for $\mathrm{GL}_n(\mathbb{K})$ and introduce basic properties of nuclear Fréchet spaces.

In Section 3, we review basic knowledge about Schwartz homology and present the propositions which are needed in our subsequent proofs. Certain homology vanishing criterion is proved in Subsection 3.3.

Section 4 is devoted to describing the orbit decomposition of the partial flag manifold under the action of Shalika subgroups. Furthermore, we calculate the conormal bundle and the modular characters which will occur in the proof of Theorem 1.2A.

The complete proof of Theorem 1.2A will be given in Section 5, where we also proved some homological finiteness results. Theorem 1.2B is proved in Section 6. In Section 7, we related twisted Shalika periods to twisted linear periods using theta correspondence. In Section 8, we study the existence of the twisted Shalika periods on the representations of $\mathrm{GL}^+_{2n}(\mathbb{R})$.

1.4. **Conventions.** The notation G_m $(m \in \mathbb{Z}_{\geq 1})$ stands for $\mathrm{GL}_m(\mathbb{K})$ through out the paper. When it is the situation that we should distinguish the case of $\mathbb{K} = \mathbb{C}$ and $\mathbb{K} = \mathbb{R}$, we shall use $\mathrm{GL}_m(\mathbb{C})$ and $\mathrm{GL}_m(\mathbb{R})$ instead.

Lie groups are denoted by capital letters, and the modular character of Lie group G is denoted by δ_G. Lie algebras of Lie groups are denoted by the corresponding Gothic letter, such as $\mathfrak{g} = \mathrm{Lie}\, G$. Denote by $\mathfrak{g}_{\mathbb{C}} := \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$ the complexified Lie algebra of G. In general, for a $\mathbb{K}$-vector space V, we use $V_{\mathbb{C}} := V \otimes_{\mathbb{R}} \mathbb{C}$ to denote its complexification.

For a subgroup $H \subset G$ and a representation π of H, take $g \in G$, we denote $H^g := g^{-1}Hg$, and π^g a representation of H^g, defined by $\pi^g(g^{-1}hg) := \pi(h)$.

For a Casselman-Wallach representation π, we write $\pi^\vee$ for the contragrediant Casselman-Wallach representation.

Topological vector spaces are assumed to be Hausdorff unless otherwise specified.

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2. Preliminaries and notation

In this section, we recall some preliminary material and fix the notation to be used in the paper.

2.1. **Local Langlands correspondence.** In this subsection, we recall basic facts about the local Langlands correspondence for $\mathrm{GL}_m(\mathbb{K})$ ($m \in \mathbb{Z}_{\geq 1}$). For a general reference, see [Kna94], [Moe97], [Jac09, Appendix].

We first review the Langlands classification of irreducible representations of $\mathrm{GL}_m(\mathbb{K})$. The characters of $\mathbb{C}^\times$ can be uniquely written as

$$\chi^{\mathbb{C}}_{k,\lambda}(z) := (\frac{z}{|z|})^k |z|^\lambda, \quad k \in \mathbb{Z},\ \lambda \in \mathbb{C},$$

the characters of $\mathbb{R}^\times$ can be uniquely written as

$$\chi^{\mathbb{R}}_{k,\lambda}(t) := \mathrm{sgn}^k(t)|t|^\lambda, \quad k \in \{0,1\},\ \lambda \in \mathbb{C}.$$

For each character $\chi^{\mathbb{K}}_{k,\lambda}$, we define its **exponent** by

$$\exp(\chi^{\mathbb{K}}_{k,\lambda}) := \mathrm{Re}\ \lambda.$$

We shall omit the superscript $\mathbb{K}$ when the context is clear and use these notations for characters of $\mathbb{K}^\times$ throughout the paper.

For each positive integer $k \in \mathbb{Z}_{\geq 1}$, we write D_k for the discrete series of $\mathrm{GL}_2(\mathbb{R})$ with central character sgn^{k+1}, whose minimal K-type has highest weight $(k+1)$. Then for $\lambda \in \mathbb{C}$, we denote

$$D_{k,\lambda} := D_k \otimes |\det_{\mathrm{GL}_2}|^\lambda.$$

The **exponent** of $D_{k,\lambda}$ is defined to be

$$\exp(D_{k,\lambda}) := \mathrm{Re}\ \lambda.$$

Denote the set of relative discrete series of $\mathrm{GL}_2(\mathbb{R})$ by

$$\mathrm{Irr}_{rd}(\mathrm{GL}_2(\mathbb{R})) := \{D_{k,\lambda}|\ k \in \mathbb{Z}_{\geq 1}, \lambda \in \mathbb{C}\}.$$

Denote by $\mathrm{Irr}(\mathrm{GL}_m(\mathbb{K}))$ the set of equivalence classes of irreducible Casselman-Wallach representations of $\mathrm{GL}_m(\mathbb{K})$. Define

$$\mathrm{Irr}^{\mathbb{K}}_{rd} := \begin{cases} \mathrm{Irr}(\mathrm{GL}_1(\mathbb{C})), & \text{if } \mathbb{K} = \mathbb{C}; \\ \mathrm{Irr}(\mathrm{GL}_1(\mathbb{R})) \amalg \mathrm{Irr}_{rd}(\mathrm{GL}_2(\mathbb{R})), & \text{if } \mathbb{K} = \mathbb{R}. \end{cases}$$

For $\pi_i \in \mathrm{Irr}^{\mathbb{K}}_{rd}$, $i = 1, \dots, r$, the corresponding normalized parabolic induced representation

$$\pi_1 \dot{\times} \cdots \dot{\times} \pi_r := \mathrm{Ind}^{G_m}_P \pi_1 \widehat{\otimes} \cdots \widehat{\otimes} \pi_r$$

is called an **generalized principal series representation**, where $P \subset G_m$ is a suitable parabolic subgroup. A generalized principal series representation is called a **induced representation of Langlands type** (or a **standard module**) if it satisfied

$$\exp(\pi_1) \geq \cdots \geq \exp(\pi_r).$$

The celebrated Langlands classification states that, for each irreducible representation π, there exists a unique standard module $\pi_1 \dot{\times} \cdots \dot{\times} \pi_r$ (up to isomorphism) such that π is the unique irreducible quotient of $\pi_1 \dot{\times} \cdots \dot{\times} \pi_r$. If this is the case, we call π the Langlands quotient of $\pi_1 \dot{\times} \cdots \dot{\times} \pi_r$, and denote

$$\pi = \pi_1 \boxplus \cdots \boxplus \pi_r.$$

Next, we recall the classification of irreducible representations of the Weil group $W_{\mathbb{K}}$ of $\mathbb{K}$. Recall that $W_{\mathbb{C}} = \mathbb{C}^\times$, and $W_{\mathbb{R}} = \mathbb{C}^\times \amalg \mathbb{C}^\times j$ with $j^2 = -1$, $jzj^{-1} = \bar{z}$ for $z \in \mathbb{C}^\times$. Since $\mathbb{C}^\times$ is abelian, the irreducible representation of $W_{\mathbb{C}}$ is just a character. As to $W_{\mathbb{R}}$, consider the reciprocity map

$$r : W_{\mathbb{R}} \to \mathbb{R}^\times$$

defined by

$$j \mapsto -1, z \mapsto z\bar{z}.$$

It is surjective, and the kernel equals the derived subgroup of $W_{\mathbb{R}}$. Thus we can view any one-dimensional representation of $W_{\mathbb{R}}$ as a character of $\mathbb{R}^{\times}$. Any irreducible representation of $W_{\mathbb{R}}$ is at most 2-dimensional. The 2-dimensional irreducible representation of $W_{\mathbb{R}}$ can be uniquely written as

$$\sigma_{k,\lambda} := \mathrm{Ind}_{\mathbb{C}^{\times}}^{W_{\mathbb{R}}} \chi_{k,2\lambda}^{\mathbb{C}}, \quad k \in \mathbb{Z}_{\geq 1},\ \lambda \in \mathbb{C}.$$

For $m \in \mathbb{Z}_{\geq 1}$, denote by $\Phi(W_{\mathbb{K}}, m)$ the set of equivalence classes of m-dimensional semisimple complex representation of $W_{\mathbb{K}}$. An element in $\Phi(W_{\mathbb{K}}, m)$ should be called an **L-parameter** for $\mathrm{GL}_m(\mathbb{K})$.

Each irreducible representation of $\mathrm{GL}_m(\mathbb{K})$ is associated with an L-parameter. Define a map L from $\mathrm{Irr}(\mathrm{GL}_m(\mathbb{K}))$ to $\Phi(W_{\mathbb{K}}, m)$ as following. For $\chi \in \mathrm{Irr}(\mathrm{GL}_1(\mathbb{K}))$, define $L(\chi) \in \Phi(W_{\mathbb{K}}, 1)$ via $\mathbb{C}^{\times} = \mathbb{C}^{\times}$ and the reciprocity map $r : W_{\mathbb{R}} \to \mathbb{R}^{\times}$. For $D_{k,\lambda} \in \mathrm{Irr}_{rd}(\mathrm{GL}_2(\mathbb{R}))$, define $L(D_{k,\lambda}) := \sigma_{k,\lambda}$. Then using the Langlands classification, we define in general

$$L(\pi_1 \boxplus \cdots \boxplus \pi_r) := L(\pi_1) \oplus \cdots \oplus L(\pi_r).$$

The local Langlands correspondence says that L is a bijection between $\mathrm{Irr}(\mathrm{GL}_m(\mathbb{K}))$ and $\Phi(W_{\mathbb{K}}, m)$, with many other favorable properties.

The following lemma describes the form for L-parameters of η-symplectic type.

Lemma 2.1. *An L-parameter $\phi : W_{\mathbb{K}} \to \mathrm{GL}_{2n}(\mathbb{C})$ is of η-symplectic type if and only if it has form*

$$\phi = \sum_{1 \leq i \leq a} \phi_i + \sum_{a+1 \leq j \leq b} (\phi_j + \phi_j^{\vee} \cdot \eta),$$

where ϕ_i $(1 \leq i \leq b)$are irreducible representations of $W_{\mathbb{K}}$, ϕ_i $(1 \leq i \leq a)$ are the two-dimensional representations of η-symplectic type, ϕ_j $(a+1 \leq j \leq b)$ aren't of η-symplectic type.

Proof. The direction of right to left is direct. As to the other direction, denote by $<,>$ the symplectic form on ϕ. Let ϕ_1 be an irreducible subrepresentation of ϕ, which isn't of η-symplectic type. Since $\dim \phi_1 \leq 2$, we have $<,> |_{\phi_1 \times \phi_1} = 0$. Let $\phi_1^{\perp} := \{v \in \phi | < v, w >= 0, \forall w \in \phi_1\}$. Since ϕ is semisimple, there exist subrepresentation ϕ_2 such that $\phi = \phi_1^{\perp} \oplus \phi_2$, also $\dim \phi_2 = \dim \phi_1$. Then $<,> |_{\phi_1 \times \phi_2}$ is non-degenerate, and $\phi_2 \cong \phi_1^{\vee} \otimes \eta$. $\square$

We further recall the following criterion for generic representations, which will be used in the proof of Theorem 1.2B.

Lemma 2.2 ([Vog78, Theorem 6.2], [FSX18, Lemma 4.8])**.** *Every irreducible generic Casselman-Wallach representation of $\mathrm{GL}_n(\mathbb{K})$ is isomorphic to its standard module.*

2.2. **Nuclear Fréchet spaces.** In this subsection, we briefly recall some standard facts about nuclear Fréchet spaces. See [CHM00, Appendix A] for more details.

Let $T : V \to W$ be a morphism of topological vector space, it is called **strict** if the induced map $V/\mathrm{Ker}T \to \mathrm{Im}\, T$ is a topological isomorphism.

Definition 2.3. *We call a sequence of (not necessarily Hausdorff) topological vector spaces*

$$\cdots \to V_{i+1} \xrightarrow{d_{i+1}} V_i \xrightarrow{d_i} V_{i-1} \to \cdots$$

*a **weak exact sequence**, if the sequence is exact as vector space and all the morphisms are continuous. Moreover, if all the morphisms are strict, we'll call the sequence a **strict exact sequence**.*

In this paper, we mainly consider nuclear Fréchet spaces, NF-spaces for short. If W is a NF-space, $V \subset W$ is a closed subspace, then V and W/V are both NF-spaces. Furthermore, all surjective morphisms between Fréchet spaces are open.

For two topological vector spaces V and W, denote by $V\widehat{\otimes}W$ the completed projective tensor product of V and W. If V and W are both NF-spaces, so is $V\widehat{\otimes}W$. The following lemma is useful in practice.

Lemma 2.4 ([CHM00, Lemma A.3])**.** *Let*

$$0 \to E_1 \xrightarrow{i} E_2 \xrightarrow{p} E_3 \to 0$$

be a strict exact sequence of NF-spaces. Let F be an NF-space. Then

$$0 \to E_1\widehat{\otimes}F \xrightarrow{i\widehat{\otimes}1_F} E_2\widehat{\otimes}F \xrightarrow{p\widehat{\otimes}1_F} E_3\widehat{\otimes}F \to 0$$

is also a strict exact sequence.

3. Schwartz homology

The theory of Schwartz homology is developed in [CS21]. In this section, we review basic knowledge about Schwartz homology and present the propositions that will be used in our subsequent proofs. We refer readers to [CS21] for more details.

3.1. **Borel's lemma.** Let M be a Nash manifold, and let $\mathcal{E}$ be a tempered vector bundle over M. Denote $\Gamma^{\mathcal{S}}(M, \mathcal{E})$ to be the Schwartz section of $\mathcal{E}$ over M, which is a Fréchet space. Suppose that U is an open Nash submanifold of M, extension by zero yields a closed embedding $\Gamma^{\mathcal{S}}(U, \mathcal{E}) \hookrightarrow \Gamma^{\mathcal{S}}(M, \mathcal{E})$.

Denote $Z := M \backslash U$ and define

$$\Gamma^{\mathcal{S}}_Z(M, \mathcal{E}) := \Gamma^{\mathcal{S}}(M, \mathcal{E})/\Gamma^{\mathcal{S}}(U, \mathcal{E}).$$

Denote by $\mathcal{N}^*_Z(M)$ the complexified conormal bundle of Z in M. We have the following description of $\Gamma^{\mathcal{S}}_Z(M, \mathcal{E})$.

Proposition 3.1 (Borel's Lemma, [AG13, Lemma B.0.8, B.0.9], [CS21, Proposition 8.2, 8.3], [Xue20, Proposition 2.5])**.** *There is a decreasing filtration on $\Gamma^{\mathcal{S}}_Z(M, \mathcal{E})$, denoted by $\Gamma^{\mathcal{S}}_Z(M, \mathcal{E})_k$, $k \in \mathbb{Z}_{\geq 0}$, such that*

$$\Gamma^{\mathcal{S}}_Z(M, \mathcal{E}) \cong \varprojlim_k \Gamma^{\mathcal{S}}_Z(M, \mathcal{E})/\Gamma^{\mathcal{S}}_Z(M, \mathcal{E})_k$$

as topological vector space, with the graded pieces

$$\Gamma^{\mathcal{S}}_Z(M, \mathcal{E})_k/\Gamma^{\mathcal{S}}_Z(M, \mathcal{E})_{k+1} \cong \Gamma^{\mathcal{S}}(Z, \mathrm{Sym}^k(\mathcal{N}^*_Z(M)) \otimes \mathcal{E}), \quad \forall k \in \mathbb{Z}_{\geq 0}.$$

Moreover, let G be an almost linear Nash group. If M is a G-Nash manifold, Z is stable under the action of G and $\mathcal{E}$ is a tempered G-bundle, then the filtration is stable under the action of G.

3.2. **Schwartz homology.** Let G be an almost linear Nash group. Denote by $\mathcal{S}\mathrm{mod}_G$ the category of smooth Fréchet representations of G of moderate growth.

For $V \in \mathcal{S}\mathrm{mod}_G$, consider the G-coinvariance $V_G := V/(\sum_{g \in G}(g-1) \cdot V)$, which is given the quotient topology and becomes a locally convex (not necessarily Hausdorff) topological vector space. The coinvariant functor $V \mapsto V_G$ is a right exact functor from $\mathcal{S}\mathrm{mod}_G$ to the category of locally convex (not necessarily Hausdorff) topological vector spaces. In [CS21], the Schwartz homology $\mathrm{H}^{\mathcal{S}}_i(G, -)$ is introduced to be the derived functor of $V \mapsto V_G$, which processes the following favorable properties.

Proposition 3.2 (Shapiro's Lemma, [CS21, Theorem 7.5]). *Let H be a Nash subgroup of an almost linear Nash group G, and $V \in \mathcal{S}\mathrm{mod}_H$. Then there is an identification*

$$\mathrm{H}_i^{\mathcal{S}}(G, (\mathrm{ind}_H^G(V \otimes \delta_H)) \otimes \delta_G^{-1}) = \mathrm{H}_i^{\mathcal{S}}(H, V), \quad \forall\, i \in \mathbb{Z},$$

of topological vector spaces, where ind_H^G denotes the Schwartz induction.

Remark. When G/H is compact, the Schwartz induction ind_H^G coincides with the unnormalized induction ${}^{un}\mathrm{Ind}_H^G$.

Proposition 3.3 ([CS21, Theorem 7.7], [ST23, Lemma 3.1]). *For every representation V in the category $\mathcal{S}\mathrm{mod}_G$, there is an identification*

$$\mathrm{H}_i^{\mathcal{S}}(G, V) = \mathrm{H}_i(\mathfrak{g}_{\mathbb{C}}, K; V) = \mathrm{H}_i(\mathfrak{g}_{\mathbb{C}}, K; V^{K\text{-}fin}), \quad \forall i \in \mathbb{Z},$$

of topological vector spaces. Here and henceforth, for $V \in \mathcal{S}\mathrm{mod}_G$, $\mathrm{H}_*(\mathfrak{g}_{\mathbb{C}}, K; V)$ denotes the homology groups of the Koszul complex

$$\cdots \to (\wedge^{l+1}(\mathfrak{g}_{\mathbb{C}}/\mathfrak{k}_{\mathbb{C}}) \otimes V)_K \to (\wedge^{l}(\mathfrak{g}_{\mathbb{C}}/\mathfrak{k}_{\mathbb{C}}) \otimes V)_K \to (\wedge^{l-1}(\mathfrak{g}_{\mathbb{C}}/\mathfrak{k}_{\mathbb{C}}) \otimes V)_K \to \cdots$$

Proposition 3.4 ([CS21, Corollary 7.8]). *Every short exact sequence*

$$0 \to V_1 \to V_2 \to V_3 \to 0$$

in the category $\mathcal{S}\mathrm{mod}_G$ yields a long exact sequence

$$\cdots \to \mathrm{H}_{i+1}^{\mathcal{S}}(G, V_3) \to \mathrm{H}_i^{\mathcal{S}}(G, V_1) \to \mathrm{H}_i^{\mathcal{S}}(G, V_2) \to \mathrm{H}_i^{\mathcal{S}}(G, V_3) \to \cdots$$

of (not necessarily Hausdorff) locally convex topological vector spaces.

In order to use Borel's lemma to study the Schwartz homology, we also need the following proposition.

Proposition 3.5 ([CS21, Lemma 8.4]). *Suppose that Z is a G-stable closed Nash submanifold of a G-Nash manifold M. Fix a character χ of G. Let $i \in \mathbb{Z}$ and assume that $\mathrm{H}_{i+1}^{\mathcal{S}}(G; (\Gamma_Z^{\varsigma}(M, \mathsf{E})/\Gamma_Z^{\varsigma}(M, \mathsf{E})_k) \otimes \chi)$ is finite-dimensional for all $k \geq 0$. Then the canonical map*

$$\mathrm{H}_i^{\mathcal{S}}(G; \Gamma_Z^{\varsigma}(M, \mathsf{E}) \otimes \chi) \to \varprojlim_k \mathrm{H}_i^{\mathcal{S}}(G; (\Gamma_Z^{\varsigma}(M, \mathsf{E})/\Gamma_Z^{\varsigma}(M, \mathsf{E})_k) \otimes \chi)$$

is a linear isomorphism.

It is important in practice to determine the Hausdorffness of the Schwartz homology. We have the following two useful propositions.

Proposition 3.6 ([CS21, Theorem 5.9, Proposition 5.7]). *Let G be an almost linear Nash group, and let V be a relatively projective representation in $\mathcal{S}\mathrm{mod}_G$. Then the coinvariant space V_G is a Fréchet space. Moreover, when G is compact, every representation in $\mathcal{S}\mathrm{mod}_G$ is relatively projective.*

Proposition 3.7 ([BW80, Lemma 3.4], [CS21, Proposition 1.9]). *Let G be an almost linear Nash group, and let $V \in \mathcal{S}\mathrm{mod}_G$. If $\mathrm{H}_i^{\mathcal{S}}(G; V)$ ($i \in \mathbb{Z}$) is finite-dimensional, then it is Hausdorff.*

Since $\mathcal{S}\mathrm{mod}_G$ is not an Abelian category, the usual homological tools cannot be applied directly. We prove the following two useful formulas for the Schwartz homology, with their proofs deferred to Appendix A.

Proposition 3.8 (Künneth formula, Theorem A.7). *Let G_1 and G_2 be two almost linear Nash groups. Assume $V_i \in \mathcal{S}\mathrm{mod}_{G_i}$, $i = 1, 2$, are both NF-spaces. If $\forall\ j \in \mathbb{Z}$, $\mathrm{H}^{\mathcal{S}}_j(G_i, V_i)$ are NF-spaces, then*

$$\mathrm{H}^{\mathcal{S}}_m(G_1 \times G_2, V_1 \widehat{\otimes} V_2) \cong \bigoplus_{p+q=m} \mathrm{H}^{\mathcal{S}}_p(G_1, V_1) \widehat{\otimes}\, \mathrm{H}^{\mathcal{S}}_q(G_2, V_2), \quad \forall m \in \mathbb{Z},$$

as topological vector spaces. In particular, $\mathrm{H}^{\mathcal{S}}_m(G_1 \times G_2, V_1 \widehat{\otimes} V_2)$ are NF-spaces.

Remark. This Künneth formula cannot be derived from that of the relative Lie algebra homology, as we do not assume the representations to be admissible.

Proposition 3.9 (Hochschild-Serre spectral sequence for nilpotent normal subgroups, Corallary A.10). *Let $H = L \ltimes N$ be an almost linear Nash group with N nilpotent normal subgroup. Consider $V \in \mathcal{S}\mathrm{mod}_H$, if $\forall\ j \in \mathbb{Z}$, $\mathrm{H}^{\mathcal{S}}_j(N, V) \in \mathcal{S}\mathrm{mod}_L$, then there exists convergent first quadrant spectral sequences:*

$$E^2_{p,q} = \mathrm{H}^{\mathcal{S}}_p(L, \mathrm{H}^{\mathcal{S}}_q(N, V)) \implies \mathrm{H}^{\mathcal{S}}_{p+q}(H, V).$$

3.3. **Some vanishing criterion.** In this section, we state two vanishing criteria for the Schwartz homology groups, which will be frequently used in the subsequent sections.

Lemma 3.10. *Let $\mathfrak{n}$ be a nilpotent complex Lie algebra, and let ψ be a non-trivial character on $\mathfrak{n}$. Denote by V a (probably infinite-dimensional) trivial representation of $\mathfrak{n}$. Then $\mathrm{H}_i(\mathfrak{n}, V \otimes \psi) = 0,\ \forall i \in \mathbb{Z}$.*

Proof. When $\dim \mathfrak{n} = 1$, then the homology group can be computed by

$$\begin{array}{cccc} 0 \to & \mathfrak{n} \otimes V \otimes \psi & \to & V \otimes \psi & \to 0 \\ & X \otimes v & \mapsto & \psi(X)v & \end{array}.$$

Since ψ is nontrivial, the middle map is a bijective map. Thus

$$\mathrm{H}_i(\mathfrak{n}, V \otimes \psi) = 0,\ \forall i \in \mathbb{Z}.$$

Now we assume $\mathfrak{n}$ is abelian, then there exist ideal $\mathfrak{n}_0 \subset \mathfrak{n}$, with $\dim \mathfrak{n}_0 = 1$, s.t. $\psi|_{\mathfrak{n}_0} \neq 0$. Thus we have $\mathrm{H}_i(\mathfrak{n}_0, V \otimes \psi) = 0,\ \forall i \in \mathbb{Z}$. Following from spectral sequence, we have $\mathrm{H}_i(\mathfrak{n}, V \otimes \psi) = 0,\ \forall i \in \mathbb{Z}$.

For a nilpotent lie algebra $\mathfrak{n}$, let $\mathfrak{n}^{(1)} := [\mathfrak{n}, \mathfrak{n}]$, then $\mathfrak{n}/\mathfrak{n}^{(1)}$ is abelian and $\psi|_{\mathfrak{n}^{(1)}} = 0$, thus ψ can be seen as a nontrivial character on $\mathfrak{n}/\mathfrak{n}^{(1)}$. Note that

$$\mathrm{H}_i(\mathfrak{n}/\mathfrak{n}^{(1)}, \mathrm{H}_j(\mathfrak{n}^{(1)}, V \otimes \psi)) \cong \mathrm{H}_i(\mathfrak{n}/\mathfrak{n}^{(1)}, \mathrm{H}_j(\mathfrak{n}^{(1)}, \mathbb{C}) \otimes V \otimes \psi)).$$

Since $\mathfrak{n}/\mathfrak{n}^{(1)}$ acts on $\mathfrak{n}^{(1)}$ algebraically, $\mathrm{H}_j(\mathfrak{n}^{(1)}, \mathbb{C})$ admits a finite filtration with each grading piece trivial action. Then using the abelian case, we have

$$\mathrm{H}_i(\mathfrak{n}/\mathfrak{n}^{(1)}, \mathrm{H}_j(\mathfrak{n}^{(1)}, V \otimes \psi)) = 0,$$

thus $\mathrm{H}_i(\mathfrak{n}, V \otimes \psi) = 0,\ \forall i \in \mathbb{Z}$. ☐

Lemma 3.11. *Let $V \in \mathcal{S}\mathrm{mod}_G$. Denote the center of G by $C(G)$. Assume $\exists\ a \in C(G)$, which acts on V by a scalar $c \neq 1$. Then $\mathrm{H}^{\mathcal{S}}_i(G, V) = 0,\ \forall\ i \in \mathbb{Z}$.*

Proof. Since $a \in C(G)$, then $\phi := \pi(a) \in \mathrm{Hom}_G(V, V)$. Also $\phi(v) = c \cdot v$, thus $\phi : \mathrm{H}^{\mathcal{S}}_i(G, V) \to \mathrm{H}^{\mathcal{S}}_i(G, V)$ maps by the scalar c. On the other hand, consider $P_{\cdot} \twoheadrightarrow V$ a relative strong projective resolution of V. Then $\pi_i(a) : P_i \to P_i$ gives a lifting of ϕ to a morphism of chain complexes. Note that $\pi_i(a) : (P_i)_G \to (P_i)_G$ descent to identity maps. Thus ϕ induce identity map on $\mathrm{H}^{\mathcal{S}}_i(G, V)$. Since $c \neq 1$, we have $\mathrm{H}^{\mathcal{S}}_i(G, V) = 0,\ \forall\ i \in \mathbb{Z}$. ☐

4. Orbit decomposition

In this section, we describe the orbit decomposition of the partial flag variety with respect to the Shalika subgroup. Furthermore, we establish certain numerical lemmas, preparing for the calculation in the following sections.

We first give some general notations on root systems. For $m \in \mathbb{Z}_{\geq 1}$, let $\mathfrak{t}_m$ be the Lie subalgebra of diagonal matrix in $\mathfrak{gl}_m$. Define e_i to be an element in $\mathfrak{t}_m^*$ by

$$e_i\left(\begin{bmatrix} x_1 & & \\ & \ddots & \\ & & x_m \end{bmatrix}\right) = x_i$$

and denote $\alpha_{i.j} := e_i - e_j$ for $1 \leq i \neq j \leq m$. Then the set of roots of $\mathfrak{gl}_m$ with respect to $\mathfrak{t}_m$ is

$$\Phi_m := \{\alpha_{i,j},\ 1 \leq i \neq j \leq m\},$$

with positive roots

$$\Phi_m^+ := \{\alpha_{i,j},\ 1 \leq i < j \leq m\}.$$

We simply denote $\alpha_i := \alpha_{i,i+1}$ for $i \in \{1, \ldots, m-1\}$, then $\Delta_m := \{\alpha_1, \alpha_2, ..., \alpha_{m-1}\}$ is the set of simple roots. For each standard parabolic subgroup $Q \in G_m$, denote by Δ_Q the subset of Δ_m corresponding to Q.

Denote W_m to be the Weyl group of $\mathfrak{gl}_m$ with respect to $\mathfrak{t}_m$, which is just the symmetric group $\mathfrak{S}_m$. We write the permutations in $\mathfrak{S}_m$ as

$$\omega = (i_1, \ldots, i_m),$$

where $i_k = \omega(k), k = 1, \ldots, m$, and identify $\mathfrak{S}_m$ with the group of $m \times m$ permutation matrices such that

$$\omega E_{j,k} \omega^{-1} = E_{i_j, i_k}, \quad j, k = 1, 2, \ldots, n,$$

where $E_{j,k}$ denotes the standard $m \times m$ elementary matrices. From now on, the size of $E_{j,k}$ will be $2n$ whenever we use this notation except in Section 7. For instance the following Weyl element ω_ι in $\mathfrak{S}_{2n}$ corresponding to the matrix

$$\omega_\iota = (n+1, n+2, \ldots, 2n, 1, 2, \ldots, n) \leftrightarrow \begin{bmatrix} 0 & 1_n \\ 1_n & 0 \end{bmatrix}.$$

For two standard parabolic subgroup P_1, P_2 of G_m, we define the following subset of W_m,

$${}^{P_1}{W_m}^{P_2} := \left\{\omega \in W_m \,:\, \omega(\Delta_{P_2}) \subset \Phi_m^+,\ \omega^{-1}(\Delta_{P_1}) \subset \Phi_m^+\right\},$$

which will occur in the relative Bruhat decomposition.

Let Q be the standard parabolic subgroup in G_{2n} corresponding to the partition (n, n), with Levi decomposition $Q = M \ltimes U$. Denote by S the Shalika subgroup of G_{2n}. Note that S is a subgroup of Q, with Levi decomposition $S = G_n^\Delta \ltimes U$. Here, G_n^Δ is the diagonal embedding of G_n into $M = G_n \times G_n$.

Proposition 4.1. *Let P be a standard parabolic subgroup in G_{2n}, and let Q be as defined above. Then we have the following bijection*

$$P\backslash G_{2n}/S \overset{1:1}{\longleftrightarrow} \Omega := \{\omega \in W_{2n} |\ \omega = \gamma \cdot \sigma, \gamma \in {}^{P}{W_{2n}}^{Q}, \sigma \in {}^{P_{1,\gamma}}{W_n}^{P_{2,\gamma}}\},$$

where for $\gamma \in W_{2n}$, $P^\gamma \cap (G_n \times G_n) = P_{1,\gamma} \times P_{2,\gamma}$. Here, we viewed W_n as a subgroup of W_{2n} by interchanging the first n variable.

Proof. Following the relative Bruhat decomposition, we have

$$P\backslash G_{2n}/Q \overset{1:1}{\longleftrightarrow} {}^{P}{W_n}^{Q}.$$

Since $\gamma \in W_{2n}$, $P^\gamma \cap (G_n \times G_n)$ is a parabolic subgroup in $G_n \times G_n$, denote it by $P^\gamma \cap (G_n \times G_n) = P_{1,\gamma} \times P_{2,\gamma}$. Then we have the following bijections

$$\begin{aligned} P^\gamma \cap Q\backslash Q/S &\overset{1:1}{\longleftrightarrow} (P^\gamma \cap (G_n \times G_n))\backslash G_n \times G_n/G_n^\Delta \\ &\overset{1:1}{\longleftrightarrow} P_{1,\gamma}\backslash G_n/P_{2,\gamma} \\ &\overset{1:1}{\longleftrightarrow} {}^{P_{1,\gamma}}W_n{}^{P_{2,\gamma}} \end{aligned}.$$

The first bijection follows from $P^\gamma \cap Q = (P^\gamma \cap M) \ltimes (P^\gamma \cap U)$, the third bijection follows from the relative Bruhat decomposition and the second bijection is induced from

$$\begin{aligned} P_{1,\gamma}\backslash G_n/P_{2,\gamma} &\to (P^\gamma \cap (G_n \times G_n))\backslash G_n \times G_n/G_n^\Delta \\ g &\mapsto \begin{bmatrix} g & 0 \\ 0 & 1_n \end{bmatrix} \end{aligned}.$$

The orbit decomposition follows from these two identifications. $\square$

Since then, for each $\omega \in \Omega$, we denote $\mathcal{O}_\omega$ the corresponding S-orbit in $P\backslash G_{2n}$.

In the rest of this section, we specialize to the case when $P \subset G_{2n}$ is a standard cuspidal parabolic subgroup corresponding to the partition $(n_1, \ldots, n_r)$ of $2n$. For $\mathbb{K} = \mathbb{C}$, we have $n_i = 1,\ \forall\ i \in \{1, \ldots, r = 2n\}$, and P is just the standard Borel subgroup in $\mathrm{GL}_{2n}(\mathbb{C})$. For $\mathbb{K} = \mathbb{R}$, we have $n_i \in \{1, 2\},\ \forall\ i \in \{1, \ldots, r\}$. Denote the Levi decomposition of P by $P = L \ltimes N$.

We shall present some specific calculations which will be used in the subsequent sections. Define the following subset of Φ_{2n}:

$$\Phi_n^{(1)} := \{\alpha_{i,j},\ 1 \le i \ne j \le n\},\ \Phi_n^{(2)} := \{\alpha_{n+i,n+j},\ 1 \le i \ne j \le n\},$$

and $\Phi_n^{(k),+} := \Phi_n^{(k)} \cap \Phi_{2n}^+,\ k = 1, 2$.

Lemma 4.2. *For $\gamma \in {}^PW_{2n}{}^Q$, $P_{1,\gamma}$ (resp. $P_{2,\gamma}$) as in Proposition 4.1 is a standard cuspidal paraboic subgroup in G_n, with corresponding simple roots $\gamma^{-1}(\Delta_P) \cap \Phi_n^{(1)} = \gamma^{-1}(\Delta_P) \cap \{\alpha_1, \ldots, \alpha_{n-1}\}$ (resp. $\gamma^{-1}(\Delta_P) \cap \Phi_n^{(2)} = \gamma^{-1}(\Delta_P) \cap \{\alpha_{n+1}, \ldots, \alpha_{2n-1}\}$).*

Proof. We prove for $P_{1,\gamma}$ in the following, the case of $P_{2,\gamma}$ is similar. Let $\Gamma := \Phi_{2n}^+ \amalg -\Delta_P$. Then the roots of $\mathfrak{p}_{1,\gamma}$ is $\Gamma' := \gamma^{-1}(\Gamma) \cap \Phi_n^{(1)}$. Note that $\gamma^{-1}(\Phi_{2n}^+) \cap \Phi_n^{(1)} = \Phi_n^{(1),+}$, thus $P_{1,\gamma}$ is a standard parabolic subgroup.

Assume $\alpha_{k,l} \in \gamma^{-1}(\Delta_P) \cap \Phi_n^{(1)}$, thus $k < l$ and $\gamma(l) = \gamma(k) + 1$. Since $\gamma(1) < \cdots < \gamma(n)$, we have $l = k + 1$. Thus $\gamma^{-1}(\Delta_P) \cap \Phi_n^{(1)} = \gamma^{-1}(\Delta_P) \cap \{\alpha_1, \ldots, \alpha_{n-1}\}$. Assume $\alpha_a, \alpha_{a+1} \in \gamma^{-1}(\Delta_P) \cap \{\alpha_1, \ldots, \alpha_{n-1}\}$ for some $1 \le a \le n-2$, then $\alpha_{\gamma(a),\gamma(a+1)}, \alpha_{\gamma(a+1),\gamma(a+2)} \in \Delta_P$, contradict with P is a cuspidal parabolic. Thus $P_{1,\gamma}$ is a standard cuspidal parabolic subgroup. $\square$

We classify the S-orbits on $P\backslash G_{2n}$ into different types and treat them separately.

Definition 4.3. *An orbit $\mathcal{O}_\omega$ is called **ψ-vanishing**, if $\xi_{\eta,\psi}|_{N^\omega \cap U}$ is non-trivial. It is called **ψ-unvanishing** otherwise. We shall say ω is ψ-vanishing (resp. ψ-unvanishing) for short.*

The following two lemmas give numerical conditions of the ψ-unvanishing orbits.

Lemma 4.4. *ω is ψ-unvanishing if and only if $\forall\ k \in \{1, \ldots, n\}$, we have $\omega(k) > \omega(n+k)$ or $\alpha_{\omega(k),\omega(n+k)} \in \Delta_P$.*

Proof. Note that $\xi_{\eta,\psi}|_{N^\omega \cap U}$ is trivial if and only if $\forall\ 1 \le i \le n, \alpha_{i,n+i} \notin \mathfrak{n}^\omega$. Then the lemma follows from the structure of the roots of $\mathfrak{n}$. $\square$

Lemma 4.5. *Assume ω is ψ-unvanishing. For $\beta \in \Delta_p$, if $\omega^{-1}(\beta) \in \{\alpha_{i,n+j}, 1 \leq i, j \leq n\}$, then $\omega^{-1}(\beta) \in \{\alpha_{i,n+j}, 1 \leq i \leq j \leq n\}$.*

Proof. Take $\beta = \alpha_a$ such that $\omega^{-1}(\alpha_a) \in \{\alpha_{i,n+j}, 1 \leq i, j \leq n\}$, thus $1 \leq \omega^{-1}(a) \leq n$, $\omega^{-1}(a+1) \geq n+1$. We prove the lemma by contradiction.

If $\omega^{-1}(a) + n > \omega^{-1}(a+1)$, then $\omega(\omega^{-1}(a)+n) = \gamma(\omega^{-1}(a)+n) > \gamma(\omega^{-1}(a+1)) = \omega(\omega^{-1}(a+1)) = a+1$. On the other hand, since ω is ψ-unvanishing, then we have $a = \omega(\omega^{-1}(a)) > \omega(\omega^{-1}(a)+n)$ or $\omega(\omega^{-1}(a)+n) = a+1$. In the first case, we obtain $a > \omega(\omega^{-1}(a)+n) > a+1$, contradicts. In the second case, we obtain $\omega^{-1}(a) + n = \omega^{-1}(a+1)$, contradicts. □

For $\omega \in \Omega$ as in Proposition 4.1, we introduce the following notation, which will be helpful for the discussion below.

- $\Psi_{P,\omega} := \omega^{-1}(\Delta_P)$,
- $\Psi^{ma}_{P,\omega} := \Psi_{P,\omega} \cap \omega_\iota(\Psi_{P,\omega})$, where $\omega_\iota = (n+1, n+2, \ldots, 2n, 1, 2, \ldots, n)$,
- $\Psi^{wh}_{P,\omega} := \Psi_{P,\omega} \cap \{\alpha_{i,n+i},\ i \in \{1, \ldots, n\}\}$,
- $\Psi^{um}_{P,\omega} := \Psi_{P,\omega} \backslash (\Psi^{ma}_{P,\omega} \amalg \Psi^{wh}_{P,\omega})$,
- $\Delta^*_{P,\omega} := \omega(\Psi^*_{P,\omega})$, for $* \in \{ma,\ wh,\ um\}$

Definition 4.6. *A ψ-unvanishing orbit $\mathcal{O}_\omega$ is called a* ***matching orbit****, if*

$$\Psi^{um}_{P,\omega} = \varnothing.$$

Otherwise, it is called an ***unmatching orbit****. We shall say ω is matching (resp. unmatching) for short.*

We introduce the following notation which will be used in the rest of the paper. Define $a_1 := 1$. For $i \in \{2, 3, \ldots, r\}$, let $a_i := 1 + \sum_{j=1}^{i-1} n_j$. If $n_i = 2$, let $b_i := a_i + 1$. Then we have $\{a_i\} \amalg \{b_i\} = \{1, \cdots, 2n\}$.

We may briefly illustrate why the matching orbits are significant. For each matching orbit $\mathcal{O}_\omega$, we can attach an element s_ω in $\mathfrak{S}_r$ as follows. If $\omega(k) = a_i$, $\omega(n+k) = a_j$, then we set $s_\omega(i) = j$, $s_\omega(j) = i$. If $\omega(k) = a_i$, $\omega(n+k) = b_i$, then we set $s_\omega(i) = i$ (in this case $n_i = 2$). Such s_ω is well-defined under the condition that ω is matching. It in fact gives a partition of $\{1, \ldots, r\}$ into pairs and singleton, which corresponds to the symplectic condition of the L-parameters.

In order to use the argument of spectral sequence, we need the following group decomposition of $P^\omega \cap S$, which is slightly different from the Levi decomposition of $P^\omega \cap S$.

Since ω is a Weyl element, we have $P^\omega \cap Q = (P^\omega \cap M) \cdot (P^\omega \cap U)$. Thus $P^\omega \cap S = (P^\omega \cap G_n^\Delta) \cdot (P^\omega \cap U)$. Denote $P^\omega \cap M = P_{1,\omega} \times P_{2,\omega}$, then $P^\omega \cap G_n^\Delta = (P_{1,\omega} \cap P_{2,\omega})^\Delta$. Denote the Levi decomposition of $P_{1,\omega} \cap P_{2,\omega} := A_\omega \ltimes B_\omega$.

Also note that $P^\omega \cap U = (L^\omega \cap U) \ltimes (N^\omega \cap U)$. Denote by $U^\dagger$ the subgroup of U generated by $\{1 + E_{i,n+i} |\ i = 1, \ldots, n\}$, split $P^\omega \cap U$ into two parts $P^\omega \cap U = (L^\omega \cap U^\dagger) \ltimes C_\omega$. Then we define $R_\omega := A^\Delta_\omega \cdot (L^\omega \cap U^\dagger)$, $V_\omega := B^\Delta_\omega \cdot C_\omega$.

Lemma 4.7. *For a ψ-unvanishing S-orbit $\mathcal{O}_\omega$, we have*

$$P^\omega \cap S = R_\omega \ltimes V_\omega.$$

Proof. It suffices to prove that $L^\omega \cap U^\dagger$ stable V_ω. Note that B_ω consists of strictly block upper triangular matrices, and $L^\omega \cap U^\dagger$ has the same block type with B_ω. Then the lemma is a direct calculation. □

The following lemma shows that if α is a simple root of $\mathfrak{p}$, then either $\omega^{-1}(\alpha)$ or $\omega^{-1}(\alpha) + \omega_\iota(\omega^{-1}(\alpha))$ is contained in roots of $\mathfrak{p}^\omega \cap \mathfrak{s}$.

Lemma 4.8. *If $\alpha_{n+k} \in \omega^{-1}(\Delta_P)$ for some $1 \le k \le n-1$, then $\omega(\alpha_k) \in \Phi_{2n}^+$.*

Proof. Denote $\omega = \gamma \cdot \sigma \in \Omega$ and $\beta := \omega(\alpha_{n+k}) \in \Delta_P$, then $\gamma^{-1}(\beta) = \alpha_{n+k}$. Note that σ satisfies $\sigma(\omega_\iota(\gamma^{-1}(\Delta_P) \cap \{\alpha_{n+1}, \dots, \alpha_{2n-1}\})) \in \Phi_n^{(1),+}$, thus $\sigma(\alpha_k) \in \Phi_n^{(1),+}$. Following the property of $\gamma \in {}^P W_{2n}{}^Q$, we obtain $\omega(\alpha_k) \in \Phi_{2n}^+$. $\square$

The idea is rather simple, although the formulation is abstract. Let us present an example in $\mathrm{GL}_6(\mathbb{R})$ to illustrate the idea. Readers are encouraged to work through this example to gain a better understanding. Note that for the case of $\mathrm{GL}_{2n}(\mathbb{C})$, we have $\Delta_P = \varnothing$. Thus matching orbits are just ψ-unvanishing S-orbits, and all the formulation above is almost trivial.

Example. Let $P = LN$ be the standard parabolic subgroup of $\mathrm{GL}_6(\mathbb{R})$ with Levi subgroup $L = \mathrm{GL}_2(\mathbb{R}) \times \mathrm{GL}_2(\mathbb{R}) \times \mathrm{GL}_2(\mathbb{R})$. There are only 5 ψ-unvanishing S-orbits, which are represented by $\{\sigma_1 = (1,3,5,2,4,6), \sigma_2 = (1,5,6,2,3,4), \sigma_3 = (3,4,5,1,2,6), \sigma_4 = (5,6,3,1,2,4), \sigma_5 = (3,5,6,1,2,4)\}$. One can check that, in this case, the only unmatching orbit is $\mathcal{O}_{\sigma_5}$, while the other four orbits are all matching orbits. Now we use σ_3 and σ_5 to show what happens on group decomposition. For the case of $\mathcal{O}_{\sigma_3}$, we have

$$P^{\sigma_3} = \begin{bmatrix} * & * & - & 0 & 0 & - \\ * & * & - & 0 & 0 & - \\ 0 & 0 & * & 0 & 0 & * \\ - & - & - & * & * & - \\ - & - & - & * & * & - \\ 0 & 0 & * & 0 & 0 & * \end{bmatrix}, \quad P^{\sigma_3} \cap S = \begin{bmatrix} a_1 & a_2 & n_1 & 0 & 0 & - \\ a_3 & a_4 & n_2 & 0 & 0 & - \\ 0 & 0 & b_1 & 0 & 0 & * \\ 0 & 0 & 0 & a_1 & a_2 & n_1 \\ 0 & 0 & 0 & a_3 & a_4 & n_2 \\ 0 & 0 & 0 & 0 & 0 & b_1 \end{bmatrix}.$$

Here and follows, both $*$ and $-$ denote arbitrary elements with $*$ contains in L^{σ_i} and $-$ contains in N^{σ_i}. The letters $a_1, \dots, b_1, \dots, n_1, \dots$ denote arbitrary elements, but the two appearances of the same letter denote the same numbers.

In this case the group decomposition in Lemma 4.7 is just

$$R_{\sigma_3} = \begin{bmatrix} a_1 & a_2 & 0 & 0 & 0 & 0 \\ a_3 & a_4 & 0 & 0 & 0 & 0 \\ 0 & 0 & b_1 & 0 & 0 & * \\ 0 & 0 & 0 & a_1 & a_2 & 0 \\ 0 & 0 & 0 & a_3 & a_4 & 0 \\ 0 & 0 & 0 & 0 & 0 & b_1 \end{bmatrix}, \quad V_{\sigma_3} = \begin{bmatrix} 1 & 0 & n_1 & 0 & 0 & - \\ 0 & 1 & n_2 & 0 & 0 & - \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & n_1 \\ 0 & 0 & 0 & 0 & 1 & n_2 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

For the unmatching orbit $\mathcal{O}_{\sigma_5}$, we have

$$P^{\sigma_5} = \begin{bmatrix} * & - & - & 0 & 0 & * \\ 0 & * & * & 0 & 0 & 0 \\ 0 & * & * & 0 & 0 & 0 \\ - & - & - & * & * & - \\ - & - & - & * & * & - \\ * & - & - & 0 & 0 & * \end{bmatrix}, \quad P^{\sigma_5} \cap S = \begin{bmatrix} a_1 & n_1 & n_2 & 0 & 0 & * \\ 0 & a_2 & n_3 & 0 & 0 & 0 \\ 0 & 0 & a_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & a_1 & n_1 & n_2 \\ 0 & 0 & 0 & 0 & a_2 & n_3 \\ 0 & 0 & 0 & 0 & 0 & a_3 \end{bmatrix}.$$

In this case the group decomposition in Lemma 4.7 is just

$$R_{\sigma_5} = \begin{bmatrix} a_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & a_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & a_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & a_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & a_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & a_3 \end{bmatrix}, \quad V_{\sigma_5} = \begin{bmatrix} 1 & n_1 & n_2 & 0 & 0 & * \\ 0 & 1 & n_3 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & n_1 & n_2 \\ 0 & 0 & 0 & 0 & 1 & n_3 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

In order to use Borel's lemma, we need the following two numerical lemmas about the representative elements of the conormal bundle.

Lemma 4.9. *Let $\mathcal{O}_\omega$ be a ψ-unvanishing orbit. Denote*

$$\begin{aligned} \mathcal{N}_\omega := \mathrm{Span}_{\mathbb{K}}(\{E_{n+j,i} | 1 \le i < j \le n,\ \omega(i) < \omega(n+j),\ \alpha_{i,n+j} \notin \omega^{-1}(\Delta_P)\} \\ \amalg \{E_{j,i} | 1 \le i < j \le n,\ \omega(i) < \omega(j),\ \alpha_{i,j} \notin \omega^{-1}(\Delta_P),\ \alpha_{n+i,n+j} \notin \omega^{-1}(\Delta_P)\}). \end{aligned}$$

Then we have

$$\mathfrak{gl}_{2n} = \mathcal{N}_\omega \oplus (\mathfrak{p}^\omega + \mathfrak{s}).$$

Proof. Denote by $\bar{\mathfrak{u}}$ (resp. $\bar{\mathfrak{n}}$) the transpose of $\mathfrak{u}$ (resp. $\mathfrak{n}$). For $i,j \in \{1,\dots,n\}$, if $E_{n+j,i} \in \mathfrak{p}^\omega + \mathfrak{s}$, then $E_{n+j,i} \in \mathfrak{p}^\omega \cap \bar{\mathfrak{u}}$. Note that $\bar{\mathfrak{u}} = \mathfrak{p}^\omega \cap \bar{\mathfrak{u}} \oplus \bar{\mathfrak{n}}^\omega \cap \bar{\mathfrak{u}}$, thus $E_{n+j,i} \notin \mathfrak{p}^\omega + \mathfrak{s} \Leftrightarrow E_{n+j,i} \in \bar{\mathfrak{n}}^\omega \cap \bar{\mathfrak{u}} \Leftrightarrow E_{i,n+j} \in \mathfrak{n}^\omega \cap \mathfrak{u}$. Since ω is ψ-unvanishing, we can directly calculate that $\mathfrak{n}^\omega \cap \mathfrak{u} = \mathrm{Span}_{\mathbb{K}}\{E_{i,n+j} | 1 \le i < j \le n, \omega(i) < \omega(n+j), \omega(\alpha_{i,n+j}) \notin \Delta_P\}$.

Since $E_{j,i} \notin p^\omega + \mathfrak{s} \Leftrightarrow E_{j,i} \notin p^\omega \cap \mathfrak{gl}_n^{(1)}$ and $E_{n+j,n+i} \notin p^\omega \cap \mathfrak{gl}_n^{(2)}$, one can directly calculate that $E_{j,i} \notin p^\omega + \mathfrak{s} \Leftrightarrow i < j, \omega(i) < \omega(j), \alpha_{i,j} \notin \omega^{-1}(\Delta_P), \alpha_{n+i,n+j} \notin \omega^{-1}(\Delta_P)$.

Then the lemma follows from the above two calculations. $\square$

Lemma 4.10. *Let $\mathcal{O}_\omega$ be a ψ-unvanishing orbit.*

(1) If $E_{n+j,i} \in \mathcal{N}_\omega$, then we have $\omega(i) < \omega(j)$, $\omega(n+i) < \omega(n+j)$.

(2) If $E_{j,i} \in \mathcal{N}_\omega$, then we have $\omega(i) < \omega(j)$, $\omega(n+i) < \omega(n+j)$.

Proof. The second assertion follows from the ψ-unvanishing condition. As to the first one, we have $\omega(i) > \omega(n+i)$ or $\omega(i)+1 = \omega(n+i)$. Since $\omega(i) < \omega(n+j)$ and $i < j$, we obtain $\omega(n+i) < \omega(n+j)$ in both case. Similar argument gives that $\omega(i) < \omega(j)$. $\square$

In the last of this section, we study the modular characters that will occur in Shapiro's lemma in the calculation of the next section. Denote

- $\Lambda_{P,\omega}^{ma} := \{1 \le i \le n | \ \alpha_i \in \Psi_{P,\omega}^{ma}\}$,
- $\Lambda_{P,\omega}^{um} := \{(i,j) | \ 1 \le i < j \le n, \ \alpha_{i,j} \text{ or } \alpha_{i,n+j} \text{ or } \alpha_{n+i,n+j} \in \Psi_{P,\omega}^{um}\}$.

Let T be the diagonal matrices subgroup of G_{2n}. Define a character on $T \cap S$ by

$$^\circ\chi_\omega\left(\begin{bmatrix} x_1 & & & & & \\ & \ddots & & & & \\ & & x_n & & & \\ & & & x_1 & & \\ & & & & \ddots & \\ & & & & & x_n \end{bmatrix}\right) := \prod_{(i,j)\in\Psi^{um}} \left(\frac{x_i}{x_j}\right),$$

and $\chi_\omega := |{}^\circ\chi_\omega|$.

Lemma 4.11. *We have $\delta_{P^\omega \cap S} = (\delta_P^\omega \cdot \chi_\omega)^{\frac{1}{2}}$ on $T \cap S$. In particular, if $\mathcal{O}_\omega$ is a matching orbit, then we have $\delta_{P^\omega \cap S} = (\delta_P^\omega)^{\frac{1}{2}}$ on R_ω*

Proof. Since the commutator subgroup of $\mathrm{GL}_n(\mathbb{K})$ is $\mathrm{SL}_n(\mathbb{K})$ and considering the Iwasawa decomposition, the second assertion follows from the first one.

For a Lie group G, denote $^\circ\delta_G(g) := \det(\mathrm{Ad}g|_{\mathfrak{g}})$, thus $\delta_G = |{}^\circ\delta_G|$. Denote

$$t = \begin{bmatrix} x_1 & & \\ & \ddots & \\ & & x_{2n} \end{bmatrix} \in T \cap S, \text{ where } x_i = x_{n+i}, \ \forall\, i \in \{1,\dots,n\}.$$

Denote $\Lambda_1 := \{(i,j) | \ 1 \le i < j \le n, \ \omega(i) < \omega(j)\}$, $\Lambda_2 := \Lambda_{P,\omega}^{ma}$, $\Lambda_3 := \{(i,j) | \ 1 \le i < j \le n, \ \omega(i) < \omega(n+j)\}$. Then we have

$$^\circ\delta_{P^\omega \cap S}(t) = \prod_{(i,j)\in\Lambda_1} \frac{x_i}{x_j} \cdot \prod_{i\in\Lambda_2} \left(\frac{x_i}{x_{i+1}}\right)^{-1} \cdot \prod_{(i,j)\in\Lambda_3} \frac{x_i}{x_j}.$$

Denote $\Lambda_4 := \{1 \le i \le 2n | \alpha_i \in \Delta_P\}$, then

$$^\circ\delta_P^\omega = \prod_{1\le i<j\le 2n} \frac{x_{\omega^{-1}(i)}}{x_{\omega^{-1}(j)}} \cdot \prod_{i\in\Lambda_4} \left(\frac{x_{\omega^{-1}(i)}}{x_{\omega^{-1}(i+1)}}\right)^{-1}.$$

By variable substitution, we obtain

$$^\circ\delta_P^\omega \cdot {}^\circ\chi_\omega(t) = \prod_{1\le\omega(i)<\omega(j)\le 2n} \frac{x_i}{x_j} \cdot \prod_{i\in\Lambda_2} \left(\frac{x_i}{x_{i+1}}\right)^{-2}.$$

Denote $\Lambda_5 := \{(i,j) | 1 \le i,j \le 2n, \omega(i) < \omega(j)\}$, and associate a partition to it by

$$\begin{aligned}\Lambda_5 = \quad & \Lambda_5^{(1)} := \{(i,j)|\ 1 \le i < j \le n,\ \omega(i) < \omega(j)\} \\ & \amalg \Lambda_5^{(2)} := \{(n+b,n+c)|\ 1 \le b,c \le n,\ \omega(n+b) < \omega(n+c)\} \\ & \amalg \Lambda_5^{(3)} := \{(i,n+c)|\ 1 \le i,c \le n,\ \omega(i) < \omega(n+c)\} \\ & \amalg \Lambda_5^{(4)} := \{(n+a,j)|\ 1 \le a,j \le n,\ \omega(n+a) < \omega(j)\}.\end{aligned}$$

Then by direct calculation,

$$\prod_{(i,j)\in\Lambda_5^{(1)}\amalg\Lambda_5^{(2)}} \frac{x_i}{x_j} = \prod_{(i,j)\in\Lambda_1} \left(\frac{x_i}{x_j}\right)^2, \qquad \prod_{(i,j)\in\Lambda_5^{(3)}\amalg\Lambda_5^{(4)}} \frac{x_i}{x_j} = \prod_{(i,j)\in\Lambda_3} \left(\frac{x_i}{x_j}\right)^2.$$

Thus $({}^\circ\delta_{P^\omega\cap S})^2(t) = {}^\circ\delta_P^\omega \cdot {}^\circ\chi_\omega(t)$ for $t \in T \cap S$. □

5. Homology of standard modules

In this section, we investigate the Schwartz homology of the standard module, and derive Theorem 1.2A in Subsection 5.3. In Subsection 5.1, we prove Lemma 5.1, 5.5, which are crucial for our application of Borel's Lemma. In Subsection 5.2, we prove Lemma 5.9, which serves as a key step in proving Theorem 1.2A.

We fix the following notation in this section. Let S be the Shalika subgroup of G_{2n}, and let P be a standard cuspidal parabolic subgroup of G_{2n} corresponding to the partition $(n_1,\dots,n_r)$, with Levi decomposition $P = L \ltimes N$. We are divided into the following two cases.

If $\mathbb{K} = \mathbb{C}$, then $n_i = 1$ for $i \in \{1,\dots,r = 2n\}$. Take $\pi_i := \chi_{k_i,\lambda_i}$ to be a character of $\mathbb{C}^\times$, where $k_i \in \mathbb{Z}$, $\lambda_i \in \mathbb{C}$.

If $\mathbb{K} = \mathbb{R}$, then $n_i \in \{1,2\}$, for $i \in \{1,\dots,r\}$. For $n_i = 1$, take $\pi_i := \chi_{k_i,\lambda_i}$ to be a character of $\mathbb{R}^\times$, where $k_i \in \{0,1\}$, $\lambda_i \in \mathbb{C}$. For $n_i = 2$, take $\pi_i := D_{k_i,\lambda_i}$ to be a relative discrete series of $\mathrm{GL}_2(\mathbb{R})$, where $k_i \in \mathbb{Z}_{\ge 1}$, $\lambda_i \in \mathbb{C}$.

In both cases, we ask $\exp(\pi_1) \ge \cdots \ge \exp(\pi_r)$ and denote $\pi := \pi_1 \widehat{\otimes} \cdots \widehat{\otimes} \pi_r$. Let $X := P\backslash G_{2n}$ be the partial flag manifold and let $\mathcal{E}$ be the tempered bundle on X associated with the standard module $\pi_1 \dot{\times} \cdots \dot{\times} \pi_r$, i.e. $\Gamma^{\mathcal{S}}(X,\mathcal{E}) \cong \pi_1 \dot{\times} \cdots \dot{\times} \pi_r$.

5.1. Homological finiteness.

As explained in Subsection 1.2, we need to prove that for each orbit, the corresponding homology group is finite-dimensional. In this section, we establish Lemma 5.1, 5.5, which will be used in the subsequent sections. Firstly, we consider the case of ψ-vanishing orbits.

Lemma 5.1. *For an ψ-vanishing S-orbit $\mathcal{O}_\omega$, we have*

$$\mathrm{H}_i^{\mathcal{S}}(P^\omega \cap S, (\delta_P^{\frac{1}{2}}\pi)^\omega \otimes \mathrm{Sym}^k(\mathcal{N}_{\omega,\mathbb{C}}^*) \otimes \delta_{P^\omega\cap S}^{-1} \otimes \xi_{\eta,\psi}^{-1}) = 0, \quad \forall\, i \in \mathbb{Z},\ k \in \mathbb{Z}_{\ge 0}.$$

Proof. Note that $N^\omega \cap U$ is a normal subgroup of $P^\omega \cap S$. Using the spectral sequence argument, it suffices to prove

$$\mathrm{H}_i^{\mathcal{S}}(N^\omega \cap U, (\delta_P^{\frac{1}{2}}\pi)^\omega \otimes \mathrm{Sym}^k(\mathcal{N}_{\omega,\mathbb{C}}^*) \otimes \delta_{P^\omega\cap S}^{-1} \otimes \xi_{\eta,\psi}^{-1}) = 0, \quad \forall\, i \in \mathbb{Z},\ k \in \mathbb{Z}_{\ge 0}.$$

Since $N^\omega \cap U$ acts on $\mathrm{Sym}^k(\mathcal{N}_{\omega,\mathbb{C}}^*)$ algebraically, we may take a finite filtration on $\mathrm{Sym}^k(\mathcal{N}_{\omega,\mathbb{C}}^*)$ such that each grading piece F_j admits a trivial action of $N^\omega \cap U$. Since $\psi|_{N^\omega\cap U}$is non-trivial, using Lemma 3.10, we obtain

$$\mathrm{H}_i^{\mathcal{S}}(N^\omega \cap U, (\delta_P^{\frac{1}{2}}\pi)^\omega \otimes F_j \otimes \delta_{P^\omega\cap S}^{-1} \otimes \xi_{\eta,\psi}^{-1}) = 0, \quad \forall\, i \in \mathbb{Z}.$$

Thus the lemma follows from the standard long exact sequence argument. □

Denote $T_2(\mathbb{R})$ (resp. $N_2(\mathbb{R})$) to be the diagonal matrices (resp. the strictly upper triangular matrices) subgroup of $\mathrm{GL}_2(\mathbb{R})$, $B_2(\mathbb{R}) := T_2(\mathbb{R}) \ltimes N_2(\mathbb{R})$. The following three lemmas concern the homology of the relative discrete series of $\mathrm{GL}_2(\mathbb{R})$.

Lemma 5.2. *Let D_k ($k \in \mathbb{Z}_{\geq 1}$) be the discrete series of $\mathrm{GL}_2(\mathbb{R})$ as in Subsection 2.1. Then as $T_2(\mathbb{R}) = \mathbb{R}^\times \times \mathbb{R}^\times$ modules,*

$$\mathrm{H}_0^{\mathcal{S}}(N_2(\mathbb{R}), D_k) = (\epsilon |\cdot|^{\frac{k+1}{2}} \boxtimes |\cdot|^{-\frac{k+1}{2}}) \oplus (\epsilon \operatorname{sgn} |\cdot|^{\frac{k+1}{2}} \boxtimes \operatorname{sgn} |\cdot|^{-\frac{k+1}{2}}),$$

where ϵ is the sign character of $\mathbb{R}^\times$ if k is even and is the trivial character otherwise. Moreover,

$$\mathrm{H}_i^{\mathcal{S}}(N_2(\mathbb{R}), D_k) = 0, \quad \forall\, i \in \mathbb{Z}_{\geq 1}.$$

Proof. Thanks to the comparison theorem for the minimal parabolic case (see [HT98, Theorem 1], [LLY21, Theorem 5.2]), we can calculate the homology using the corresponding $(\mathfrak{g}, K)$-modules. Then the second equality follows from [CO78, Corollary 2.6] and [KV16, Corollary 3.6]. As to the first equality, it's a direct calculation that $\dim \mathrm{H}_0^{\mathcal{S}}(N_2(\mathbb{R}), D_k) = 2$, then the assertion follows from Frobenius reciprocity. □

Lemma 5.3. *Let V be a relative discrete series of $\mathrm{GL}_2(\mathbb{R})$. Denote by ψ a non-trivial unitary character on $N_2(\mathbb{R})$. Then*

$$\mathrm{H}_i^{\mathcal{S}}(N_2(\mathbb{R}), V \otimes \psi) = \begin{cases} \mathbb{C}, & \text{if } i = 0; \\ 0, & \text{otherwise.} \end{cases}$$

Proof. Recall that we have a standard short exact sequence of representation of $\mathrm{GL}_2(\mathbb{R})$,

$$0 \to F \to I \to V \to 0$$

where I is a principal series representation and F is finite-dimensional. Thus the lemma follows from [CHM00, Lemma 8.5] and the fact that $\mathrm{H}_0^{\mathcal{S}}(N_2(\mathbb{R}), F \otimes \psi) = 0$. □

Lemma 5.4. *Let V_1, V_2 be relative discrete series of $\mathrm{GL}_2(\mathbb{R})$, and let E be a finite-dimensional representation of $\mathrm{GL}_2(\mathbb{R})$, then we have,*

$$\dim \mathrm{H}_i^{\mathcal{S}}(\mathrm{GL}_2(\mathbb{R}), V_1 \widehat{\otimes} V_2 \otimes E) < \infty, \quad \forall\, i \in \mathbb{Z}.$$

Proof. Consider the standard exact sequence for V_j ($j = 1, 2$) as in Lemma 5.3,

$$0 \to F_j \to I_j \to V_j \to 0.$$

Then the lemma follows from the standard arguments of the long exact sequence. □

Denote $I_\omega^{um} := \{i \in \{1, \ldots, r\} | \alpha_{a_i} \in \Delta_{P,\omega}^{um}\}$. We shall split π into two parts. Let $\pi_{um,\omega} := \widehat{\otimes}_{i \in I_\omega^{um}} \pi_i$ and let $\pi_{ma,\omega}$ be the remain part such that $\pi = \pi_{um,\omega} \widehat{\otimes} \pi_{ma,\omega}$. When the context is clear, we will omit the subscript ω for short.

Lemma 5.5. *For an ψ-unvanishing S-orbit $\mathcal{O}_\omega$, we have*

$$\dim \mathrm{H}_i^{\mathcal{S}}(P^\omega \cap S, (\delta_P^{\frac{1}{2}} \pi)^\omega \otimes \mathrm{Sym}^k(\mathcal{N}_{\omega,\mathbb{C}}^*) \otimes \delta_{P^\omega \cap S}^{-1} \otimes \xi_{\eta,\psi}^{-1}) < \infty, \quad \forall\, i \in \mathbb{Z},\ k \in \mathbb{Z}_{\geq 0}.$$

Proof. For $\mathbb{K} = \mathbb{C}$, the lemma is obvious, since the representation occurring in the homology is already finite-dimensional. We consider the case for $\mathbb{K} = \mathbb{R}$ in the following.

Recall that in Lemma 4.7 we have decomposition $P^\omega \cap S = R_\omega \ltimes V_\omega$. Following the spectral sequence, it suffices to prove that

$$\dim \mathrm{H}_i^{\mathcal{S}}(R_\omega, \mathrm{H}_j^{\mathcal{S}}(V_\omega, (\delta_P^{\frac{1}{2}} \pi)^\omega \otimes \mathrm{Sym}^k(\mathcal{N}_{\omega,\mathbb{C}}^*) \otimes \delta_{P^\omega \cap S}^{-1} \otimes \xi_{\eta,\psi}^{-1})) < \infty, \quad \forall\, i, j \in \mathbb{Z}.$$

Since V_ω acts trivially on π^ω_{ma}, we have

$$\begin{aligned}&\mathrm{H}^{\mathcal{S}}_i(R_\omega, \mathrm{H}^{\mathcal{S}}_j(V_\omega, (\delta_P^{\frac{1}{2}}\pi)^\omega \otimes \mathrm{Sym}^k(\mathcal{N}^*_{\omega,\mathbb{C}}) \otimes \delta^{-1}_{P^\omega\cap S} \otimes \xi^{-1}_{\eta,\psi}))\\ =&\mathrm{H}^{\mathcal{S}}_i(R_\omega, \mathrm{H}^{\mathcal{S}}_j(V_\omega, \pi^\omega_{um})\widehat{\otimes}\pi^\omega_{ma} \otimes \mathrm{Sym}^k(\mathcal{N}^*_{\omega,\mathbb{C}}) \otimes (\delta_P^{\frac{1}{2}})^\omega \otimes \delta^{-1}_{P^\omega\cap S} \otimes \xi^{-1}_{\eta,\psi})\end{aligned}$$

Note that $\mathfrak{v}_\omega$ admits a filtration such that each grading piece isomorphic to $\mathfrak{n}_2(\mathbb{R})$. Thus following from Lemma 4.8, 5.2, we know that $\dim \mathrm{H}^{\mathcal{S}}_j(V_\omega, \pi^\omega_{um}) < \infty$. Note that R_ω is product of copies of $\mathrm{GL}_1(\mathbb{R})$, $\mathrm{GL}_2(\mathbb{R})$ (whose roots corresponding to $\Phi^{ma}_{P,\omega}$) and $\mathrm{GL}^\Delta_1(\mathbb{R}) \ltimes N_2(\mathbb{R}) \subset B_2(\mathbb{R})$ (whose roots corresponding to $\Phi^{wh}_{P,\omega}$). Thus the lemma follows from the Künneth formula and Lemma 5.3, 5.4. □

5.2. **Homological vanishing for unmatching orbits and normal derivative.** In this section, we prove Lemma 5.9, which is crucial for proving 1.2A. We further introduce some notations in order to provide proof. We fix a ψ-unvanishing orbit $\mathcal{O}\omega$.

Definition 5.6. *For $1 \le i < j \le n$, we say i is* ***weightly related*** *to j if $(i, j) \in \Lambda^{um}_{P,\omega}$, where $\Lambda^{um}_{P,\omega}$ is defined above Lemma 4.11. We say i is* ***weightly associated*** *to j if there is a chain $i = i_0 < i_1 < \cdots < i_l < i_{l+1} = j$, such that i_k is weightly related to i_{k+1}, $\forall\ k \in \{0, \dots, l\}$, denoted by $i \rightsquigarrow j$.*

Definition 5.7. *For $1 \le i < j \le n$, we say i is* ***derivatively related*** *to j if $E_{j,i}$ or $E_{n+j,i} \in \mathcal{N}_\omega$. We say i is* ***derivatively associated*** *to j if there is a chain $i = i_0 < i_1 < \cdots < i_l < i_{l+1} = j$, such that i_k is derivatively related to i_{k+1}, $\forall\ k \in \{0, \dots, l\}$, denoted by $i \rightarrowtail j$.*

For a character $\chi_{k,\lambda}$ of $\mathbb{K}^\times$, we say it is **positive** (resp. **negative**, **non-negative**, **non-positive**), if $\mathrm{Re}\lambda > 0$ (resp. $\mathrm{Re}\lambda < 0$, $\mathrm{Re}\lambda \ge 0$, $\mathrm{Re}\lambda \le 0$).

Lemma 5.8. *For $1 \le i < j \le n$.*
(i) If $i \rightsquigarrow j$, then $\chi^{ex}_{\omega(i)} \cdot \chi^{ex}_{\omega(n+i)} \cdot (\chi^{ex}_{\omega(j)})^{-1} \cdot (\chi^{ex}_{\omega(n+j)})^{-1}$ is non-negative.
(ii) If $i \rightarrowtail j$, then $\chi^{ex}_{\omega(i)} \cdot \chi^{ex}_{\omega(n+i)} \cdot (\chi^{ex}_{\omega(j)})^{-1} \cdot (\chi^{ex}_{\omega(n+j)})^{-1}$ is non-negative.

Proof. It suffices to prove for the case when i, j is related. Thus the second assertion follows from Lemma 4.10.

As to the first assertion, for $1 \le i < j \le n$, if $\alpha_{i,j}, \alpha_{n+i,n+j} \in \omega^{-1}(\Delta_P)$, since $\omega \in \Omega$, we obtain $\omega(i) < \omega(j)$, $\omega(n+i) < \omega(n+j)$. If $\alpha_{i,n+j} \in \omega^{-1}(\Delta_P)$, then $\omega(n+j) = \omega(i) + 1$. Thus $\omega(j) > \omega(n+j)$, $\omega(i) > \omega(n+i)$ follows from the ψ-unvanishing condition, thus $\omega(n+i) < \omega(i) < \omega(n+j) < \omega(j)$. □

Denote $J^{ma}_\omega := \{i \in \{1, \dots, n-1\} | \alpha_i \in \Psi^{ma}_{P,\omega}\}$. The following lemma is crucial for the proof of Theorem 1.2A.

Lemma 5.9. *If $k > 0$ or $\mathcal{O}_\omega$ is an unmatching orbit, then*

$$\mathrm{H}^{\mathcal{S}}_0(P^\omega \cap S, (\delta_P^{\frac{1}{2}}\pi)^\omega \otimes \mathrm{Sym}^k(\mathcal{N}^*_{\omega,\mathbb{C}}) \otimes \delta^{-1}_{P^\omega\cap S} \otimes \xi^{-1}_{\eta,\psi}) = 0.$$

Proof. We first prove the case for $\mathbb{K} = \mathbb{R}$. The case for $\mathbb{K} = \mathbb{C}$ is similar but simpler, we shall give a comment at last. Now we let $\mathbb{K} = \mathbb{R}$.

We introduce the following notation in order to prove the theorem. Recall we have defined a sequence of number $\{a_i, b_i\}$ under Definition 4.6 such that $\{a_i\} \amalg \{b_i\} = \{1, \cdots, 2n\}$. Define $\chi^{ex}_{a_i} := |\cdot|^{\lambda_i}$, $\chi^{ex}_{b_i} := |\cdot|^{\lambda_i}$. If $i \in I^{um}_\omega$, then $n_i = 2$, we define $\chi^{wt}_{a_i} := |\cdot|^{\frac{k_i}{2}}$, $\chi^{wt}_{b_i} := |\cdot|^{-\frac{k_i}{2}}$. For other $k \in \{1, \dots, 2n\}$, define $\chi^{wt}_k := \mathbb{C}$. Let $\chi_i := \chi^{ex}_i \cdot \chi^{wt}_{a_i}$, $i \in \{1, \dots, 2n\}$. We shall call χ^{ex}_i (resp. χ^{ex}_i) the exponent (resp. weight) part of χ_i.

Note that $P^\omega \cap S = R_\omega \ltimes V_\omega$, and V_ω acts algebraically on $\mathrm{Sym}^k(\mathcal{N}^*_{\omega,\mathbb{C}})$, we may take a finite filtration on $\mathrm{Sym}^k(\mathcal{N}^*_{\omega,\mathbb{C}})$ such that each grading piece W is an irreducible R_ω module with trivial V_ω action. Thus we only need to show

$$\mathrm{H}_0^{\mathcal{S}}(P^\omega \cap S, (\delta_P^{\frac{1}{2}}\pi)^\omega \otimes W \otimes \delta^{-1}_{P^\omega\cap S} \otimes \xi^{-1}_{\eta,\psi}) = 0. \tag{5.1}$$

We first take coinvariance with respect to V_ω, then we get

$$\begin{aligned} & \mathrm{H}_0^{\mathcal{S}}(P^\omega \cap S, (\delta_P^{\frac{1}{2}}\pi)^\omega \otimes W \otimes \delta^{-1}_{P^\omega\cap S} \otimes \xi^{-1}_{\eta,\psi}) \\ = \; & \mathrm{H}_0^{\mathcal{S}}(R_\omega, \mathrm{H}_0^{\mathcal{S}}(V_\omega, \pi^\omega_{um})\widehat{\otimes}\pi^\omega_{ma} \otimes W \otimes (\delta_P^{\frac{1}{2}})^\omega \otimes \delta^{-1}_{P^\omega\cap S} \otimes \xi^{-1}_{\eta,\psi}). \end{aligned}$$

As in the proof of Lemma 5.5, $\mathrm{H}_0^{\mathcal{S}}(V_\omega, \pi^\omega_{um})$ is tensor product of the Jacquet module of each π_i occuring in π_{um}. Write $E := \mathrm{H}_0^{\mathcal{S}}(V_\omega, \pi^\omega_{um})$ for short. Denote by $C(R_\omega)$ the center of R_ω, and let $C(R_\omega)^+ \subset C(R_\omega)$ be the subgroup of the matrices with each entry positive real number. Using Lemma 5.2 and Lemma 4.11, we know that $C(R_\omega)^+$ acts on

$$E \otimes \pi^\omega_{ma} \otimes W \otimes (\delta_P^{\frac{1}{2}})^\omega \otimes \delta^{-1}_{P^\omega\cap S} \otimes \xi^{-1}_{\eta,\psi}$$

as the character

$$x = \begin{bmatrix} x_1 & & & & & \\ & \ddots & & & & \\ & & x_n & & & \\ & & & x_1 & & \\ & & & & \ddots & \\ & & & & & x_n \end{bmatrix} \mapsto \alpha(x) \cdot \prod_{i=1}^n \chi_{\omega(i)}(x_i) \cdot \chi_{\omega(n+i)}(x_i) \cdot \eta^{-1}(x_i),$$

where α is the central character of W. According to Lemma 3.11, in order to prove (5.1), it suffices to show that this character is nontrivial. We use the method of infinite descent.

We first deal with the case when $k > 0$. Let α be the central character of W with

$$\alpha\left(\begin{bmatrix} x_1 & & & & & \\ & \ddots & & & & \\ & & x_n & & & \\ & & & x_1 & & \\ & & & & \ddots & \\ & & & & & x_n \end{bmatrix}\right) = \prod_{i=1}^n x_i^{r_i}.$$

Then α is one of the characters of the adjoint action of $C(R_\omega)^+$ on $\mathrm{Sym}^k(\mathcal{N}^*_{\omega,\mathbb{C}})$. Thus we have $\sum r_i = 0,\ r_i \in \mathbb{Z}$. For $i \in J^{ma}_\omega$, let $t_i = t_{i+1} := \frac{r_i + r_{i+1}}{2}$. For other $i \in \{1, \dots, n\}$, let $t_i := r_i$.

If the action of $C(R_\omega)^+$ is trivial, then we have

$$\chi_{\omega(i)}(x) \cdot \chi_{\omega(n+i)}(x) \cdot \eta^{-1}(x) \cdot x^{t_i} = 1, \quad \forall\, x \in \mathbb{R}^\times,\ i \in \{1, \dots, n\}. \tag{5.2}$$

Denote $a := \min\{i \in \{1, \dots, n\} | t_i \neq 0,\ r_i \neq 0\}$, then $t_a > 0$ since the form of elements in $\mathcal{N}_\omega$. Denote $\Omega_a := \{i = 1, \dots, n | a \rightarrowtail i\}$, and let $b := \max\{i \in \Omega_a | t_i \neq 0\}$, then we can easily deduced that $t_b < 0$, also $a < b$.

If either $\chi^{wt}_{\omega(a)}$ or $\chi^{wt}_{\omega(n+a)}$ is negative, then by definition, there exist $a' \rightsquigarrow a$ such that both $\chi^{wt}_{\omega(a')}$ and $\chi^{wt}_{\omega(n+a')}$ are non-negative, with at least one positive, also we have $t_{a'} = 0$. If $\chi^{wt}_{\omega(a)}$ and $\chi^{wt}_{\omega(n+a)}$ are both non-negative, then we let $a' = a$. We have two possible cases corresponding to these two: (i). $t_{a'} = 0$ and $\chi^{wt}_{\omega(a')} \cdot \chi^{wt}_{\omega(n+a')}$ is positive, (ii). $t_{a'} > 0$ and $\chi^{wt}_{\omega(a')} \cdot \chi^{wt}_{\omega(n+a')}$ is non-negative.

Now, if $\chi^{wt}_{\omega(b)}$ and $\chi^{wt}_{\omega(n+b)}$ are both non-positive, then $\chi_{\omega(a')} \cdot \chi_{\omega(n+a')} \cdot \chi^{-1}_{\omega(b)} \cdot \chi^{-1}_{\omega(n+b)} \cdot x^{t_{a'} - t_b}$ has non-negative exponent part (according to Lemma 5.8), positive derivative part (i.e. $t_{a'} - t_b > 0$) and non-negative weight part, contradict with the identity (5.2). Thus at least one of $\chi^{wt}_{\omega(b)}$ and $\chi^{wt}_{\omega(n+b)}$ is positive. Thus there exist $c > b,\ b \rightsquigarrow c$ such that both $\chi^{wt}_{\omega(c)}$ and $\chi^{wt}_{\omega(n+c)}$ are non-positive, with at least one negative. If $t_c \leq 0$, then

$\chi_{\omega(a')} \cdot \chi_{\omega(n+a')} \cdot \chi_{\omega(c)}^{-1} \cdot \chi_{\omega(n+c)}^{-1} \cdot x^{t_{a'}-t_c}$ has non-negative derivative part (i.e. $t_{a'} - t_c \geq 0$), positive weight part and non-negative exponent part, contradict with the identity (5.2), thus $t_c > 0$. If $r_c \neq 0$, denote $c' = c$. If $r_c = 0$, then $(c-1) \in J_\omega^{ma}$ or $c \in J_\omega^{ma}$, thus there exist $c' \in \{c+1, c-1\}$ such that $r_{c'} > 0$. In all the case we have $r_{c'} > 0$ and $\Omega_{c'} \neq \varnothing$. Take $d := \max\{i \in \Omega_{c'} | t_i \neq 0\}$, then $t_d < 0$. If $\chi_{\omega(d)}^{wt}$ and $\chi_{\omega(n+d)}^{wt}$ are both non-positive, then $\chi_{\omega(a')} \cdot \chi_{\omega(n+a')} \cdot \chi_{\omega(d)}^{-1} \cdot \chi_{\omega(n+d)}^{-1} \cdot x^{t_{a'}-t_d}$ has positive derivative part (i.e. $t_{a'} - t_d > 0$), non-negative weight part and non-negative exponent part, contradict with the identity (5.2). Thus we find $d > c' \geq b$, such that at least one of $\chi_{\omega(d)}^{wt}$ and $\chi_{\omega(n+d)}^{wt}$ is positive. Since n is finite, using the method of infinite descent, we find the contradiction.

Now we address the case where $k = 0$ and the orbits are unmatching. Similarly, we assume that the central character of $C(R_\omega)^+$ is trivial, i.e. we have

$$\chi_{\omega(i)} \cdot \chi_{\omega(n+i)} \cdot \eta^{-1} = id, \ \forall \ i \in \{1, \ldots, n\}. \tag{5.3}$$

Take $j := \max\{i \in \{1, \ldots, n\} |$ at least one of $\chi_{\omega(j)}^{wt}$ and $\chi_{\omega(n+j)}^{wt}$ is negative$\}$, which is well-defined since $\mathcal{O}_\omega$ is an unmatching orbit. Thus both $\chi_{\omega(j)}^{wt}$ and $\chi_{\omega(n+j)}^{wt}$ are non-positive. Let $a := \min\{i \in \{1, \ldots, n\} | i \rightsquigarrow j\}$, thus both $\chi_{\omega(a)}^{wt}$ and $\chi_{\omega(n+a)}^{wt}$ are non-negative. Thus $\chi_{\omega(a)} \cdot \chi_{\omega(n+a)} \cdot \chi_{\omega(j)}^{-1} \cdot \chi_{\omega(n+j)}^{-1}$ has positive weight part and non-negative exponent part, contradict with the identity (5.3).

As to the case where $\mathbb{K} = \mathbb{C}$, P is just the Borel subgroup, and the standard module is a principal series representation. The main difference lies in the part concerning the complexified conormal bundle. In this case $\mathcal{N}_{\omega,\mathbb{C}} \cong \mathcal{N}_\omega \oplus \mathcal{N}_\omega$ as $\mathbb{C}$-vector spaces, where $z \in \mathbb{C}^\times$ will act by z and $\bar{z}$ separately. However, since $|z| = |\bar{z}|$, the argument as above also works. □

5.3. **Proof of Theorem 1.2A.**

Theorem 5.10. *The notations are as introduced at the beginning of this section. Recall that the set Ω has been defined in Proposition 4.1, we have*

$$\dim \mathrm{H}_0^{\mathcal{S}}(S, \Gamma^\varsigma(X, \mathcal{E}) \otimes \xi_{\eta,\psi}^{-1}) \leq \sum_{\{\omega \in \Omega | \omega \text{ is matching}\}} \dim \mathrm{H}_0^{\mathcal{S}}(S, \Gamma^\varsigma(\mathcal{O}_\omega, \mathcal{E}) \otimes \xi_{\eta,\psi}^{-1}),$$

where the right hand side is finite.

Proof. Following the S-orbit decomposition on X as in Proposition 4.1, X admits a finite decreasing sequence of open submanifolds

$$U_1 := X \supsetneq U_2 \supsetneq \ldots \supsetneq U_f \supsetneq U_{f+1} := \varnothing,$$

such that for $i \in \{1, \ldots, f\}$, $U_i \backslash U_{i+1}$ is a S-orbit in X. When $U_i \backslash U_{i+1} = \mathcal{O}_\omega$, we shall also denote $U_\omega := U_i$. Consider the short exact sequence

$$0 \to \Gamma^\varsigma(U_{i+1}, \mathcal{E}) \otimes \xi_{\eta,\psi}^{-1} \to \Gamma^\varsigma(U_i, \mathcal{E}) \otimes \xi_{\eta,\psi}^{-1} \to \Gamma_{O_i}^\varsigma(U_i, \mathcal{E}) \otimes \xi_{\eta,\psi}^{-1} \to 0,$$

then the associated long exact sequence shows that

$$\dim \ \mathrm{H}_0^{\mathcal{S}}(S, \Gamma^\varsigma(X, \mathcal{E}) \otimes \xi_{\eta,\psi}^{-1}) \leq \sum_{\omega \in \Omega} \dim \ \mathrm{H}_0^{\mathcal{S}}(S, \Gamma_{O_\omega}^\varsigma(U_\omega, \mathcal{E}) \otimes \xi_{\eta,\psi}^{-1}).$$

According to Borel's lemma, Shapiro's lemma, and Proposition 3.5, in order to address the right hand side of the inequality, it suffices to concern with

$$\mathrm{H}_i^{\mathcal{S}}(P^\omega \cap S, (\delta_P^{\frac{1}{2}} \pi)^\omega \otimes \mathrm{Sym}^k(\mathcal{N}_{\omega,\mathbb{C}}^*) \otimes \delta_{P^\omega \cap S}^{-1} \otimes \xi_{\eta,\psi}^{-1}),$$

which has been calculated in Lemma 5.5, 5.1, 5.9. By combining these three lemmas, we obtain that

$$\mathrm{H}_0^{\mathcal{S}}(S, \Gamma^{\varsigma}_{\mathcal{O}_\omega}(U_\omega, \mathcal{E}) \otimes \xi_{\eta,\psi}^{-1}) \cong \begin{cases} \mathrm{H}_0^{\mathcal{S}}(S, \Gamma^{\varsigma}(\mathcal{O}_\omega, \mathcal{E}) \otimes \xi_{\eta,\psi}^{-1}), & \text{if } \mathcal{O}_\omega \text{ is a matching orbit;} \\ 0, & \text{otherwise,} \end{cases}$$

and thus the theorem. The finiteness of the right hand side follows from Lemma 5.3, 5.4 and the definition of the matching orbits. $\square$

Corollary 5.11. *For an irreducible Casselman-Wallach representation π of G_{2n}, we have*

$$\dim\ \mathrm{H}_0^{\mathcal{S}}(S, \pi \otimes \xi_{\eta,\psi}^{-1}) < \infty.$$

Proof. It suffices to show that $\dim\ \mathrm{H}_0^{\mathcal{S}}(S, \sigma \otimes \xi_{\eta,\psi}^{-1}) < \infty$ for each standard module σ, which follows from Theorem 5.10. $\square$

Following Theorem 5.10, it is important to study the homology of the Schwartz section over the matching orbits, which will be shown in the Proposition 5.12.

Proposition 5.12. *The notations are as introduced at the beginning of this section. If there exists a matching orbit $\mathcal{O}_\omega$ such that $\dim \mathrm{H}_0^{\mathcal{S}}(S, \Gamma^{\varsigma}(\mathcal{O}_\omega, \mathcal{E}) \otimes \xi_{\eta,\psi}^{-1}) \neq 0$, then the L-parameter of the Langlands quotient of $\pi_1 \dot{\times} \cdots \dot{\times} \pi_r$ is of η-symplectic type.*

Proof. Recall we have decomposition $P^\omega \cap S = R_\omega \ltimes V_\omega$. When $\mathcal{O}_\omega$ is a matching orbit, we know that V_ω acts trivially on the representation, thus we have

$$\begin{aligned} \mathrm{H}_0^{\mathcal{S}}(S, \Gamma^{\varsigma}(\mathcal{O}_\omega, \mathcal{E}) \otimes \xi_{\eta,\psi}^{-1}) &= \mathrm{H}_0^{\mathcal{S}}(P^\omega \cap S, (\delta_P^{\frac{1}{2}}\pi)^\omega \otimes \delta_{P^\omega \cap S}^{-1} \otimes \xi_{\eta,\psi}^{-1}) \\ &= \mathrm{H}_0^{\mathcal{S}}(R_\omega, (\delta_P^{\frac{1}{2}}\pi)^\omega \otimes \delta_{P^\omega \cap S}^{-1} \otimes \xi_{\eta,\psi}^{-1}) \\ &= \mathrm{H}_0^{\mathcal{S}}(R_\omega, \pi^\omega \otimes \xi_{\eta,\psi}^{-1}) \end{aligned},$$

where the first equality follows from Shapiro's lemma and the third follows from Lemma 4.11. Therefore, as discussed in the proof of Lemma 5.5, the theorem follows from Künneth formula, Lemma 5.3, 5.4 and [ST23, Lemma 3.9]. In fact, recall that we have defined s_ω after Definition 4.6. In this case we have $\pi_i \cong \pi_{s_\omega(i)}^\vee \cdot \eta$. $\square$

Proof of Theorem 1.2A. Let τ be an irreducible Casselman-Wallach representation of G_{2n}. If τ admits an (η, ψ)-twisted Shalika functional, then so is its standard module $\pi_1 \dot{\times} \cdots \dot{\times} \pi_r$. Let X be the corresponding partial flag manifold, and let $\mathcal{E}$ be the corresponding tempered vector bundle on X, such that $\Gamma^{\mathcal{S}}(X, \mathcal{E}) \cong \pi_1 \dot{\times} \cdots \dot{\times} \pi_r$. Thus $\dim\ \mathrm{H}_0^{\mathcal{S}}(S, \Gamma^{\varsigma}(X, \mathcal{E}) \otimes \xi_{\eta,\psi}^{-1}) \neq 0$. Then Theorem 1.2A follows from Theorem 5.10 and Proposition 5.12. $\square$

6. Proof of Theorem 1.2B

We first recall the following theorem, which constructs twisted Shalika functionals for parabolic induced representations.

Theorem 6.1 ([CJLT20, Theorem 2.1])**.** *For two even positive integers n_1 and n_2, take two Casselman-Wallach representations π_1 and π_2 of $\mathrm{GL}_{n_1}(\mathbb{K})$ and $\mathrm{GL}_{n_2}(\mathbb{K})$, respectively, and assume that both π_1 and π_2 have non-zero (η, ψ)-twisted Shalika periods. Then the normalized parabolic induction $\pi_1 \dot{\times} \pi_2$ also has a non-zero (η, ψ)-twisted Shalika periods.*

Recall in Lemma 2.1, if an L-parameter is of η-symplectic type, then it has form

$$\sum_i \phi_i + \sum_j (\phi_j + \phi_j^\vee \cdot \eta).$$

Note that for the $\mathrm{GL}_2(\mathbb{K})$ case, an irreducible representation π has an (η,ψ)-Shalika period if and only if it has a Whittaker period and the central character of π equals to η. Thus, according to Theorem 6.1, we can reduce the theorem to the following case in $\mathrm{GL}_4(\mathbb{R})$.

For a character η of $\mathbb{K}^\times$, we write $\eta_{\mathrm{GL}_m(\mathbb{K})} := \eta\circ\det_{\mathrm{GL}_m(\mathbb{K})}$ for the associated character of $\mathrm{GL}_m(\mathbb{K})$. Denote by S_{2n} the Shalika subgroup of G_{2n}.

Theorem 6.2. *Let $P\subset\mathrm{GL}_4(\mathbb{R})$ be the standard parabolic subgroup corresponding to the partition $(2,2)$. Let $\eta:\mathbb{R}^\times\to\mathbb{C}^\times,\ t\mapsto|t|^{z_0}(\operatorname{sgn} t)^{m_0},\ z_0\in\mathbb{C}^\times,\ m_0\in\{0,1\}$ be a character of $\mathbb{R}^\times$. Assume that $D_{k,\lambda}$ ($k\in\mathbb{Z}_{\geq 1},\ \lambda\in\mathbb{C}$) doesn't have a non-zero (η,ψ)-twisted Shalika peroids (thus so is $D_{k,z_0-\lambda}$), and $D_{k,\lambda}\dot\times D_{k,z_0-\lambda}$ is irreducible, then $D_{k,\lambda}\dot\times D_{k,z_0-\lambda}$ has a non-zero (η,ψ)-twisted Shalika peroids. Moreover, we have*

$$\mathrm{H}_i^{\mathcal{S}}(S_4, D_{k,\lambda}\dot\times D_{k,z_0-\lambda}\otimes\xi_{\eta,\psi}^{-1})\cong\mathrm{H}_i^{\mathcal{S}}(\mathrm{GL}_2(\mathbb{R}), D_{k,\lambda}\widehat\otimes D_{k,z_0-\lambda}\otimes\eta_{\mathrm{GL}_2(\mathbb{R})}^{-1}),\quad i\in\mathbb{Z},$$

and

$$\mathrm{Hom}_{S_4}(D_{k,\lambda}\dot\times D_{k,z_0-\lambda},\xi_{\eta,\psi})\cong\mathrm{Hom}_{\mathrm{GL}_2(\mathbb{R})}(D_{k,\lambda}\widehat\otimes D_{k,z_0-\lambda},\eta_{\mathrm{GL}_2(\mathbb{R})}).$$

Proof. According to Lemma 5.4, the second equality follows from the first one (for $i=0$) by taking continuous dual. In the following, we aim to prove the first equality.

A quick calculation shows that there are only 4 S_4-orbits on $X := P\backslash\mathrm{GL}_4(\mathbb{R})$, represented by $\{\sigma_1=(1,2,3,4),\sigma_2=(1,3,2,4),\sigma_3=(3,1,2,4),\sigma_4=(3,4,1,2)\}$, where $\mathcal{O}_{\sigma_4}$ is the unique open S_4-orbit. One can easily verify that $\mathcal{O}_{\sigma_1}$ and $\mathcal{O}_{\sigma_3}$ are ψ-vanishing, $\mathcal{O}_{\sigma_2}$ and $\mathcal{O}_{\sigma_4}$ are matching orbits. Note that $P^{\sigma_4}\cap S_4\cong\mathrm{GL}_2(\mathbb{R})$. Denote by $\mathcal{E}$ the tempered bundle on X associated with $D_{k,\lambda}\dot\times D_{k,z_0-\lambda}$. Following the Shapiro's lemma and Lemma 4.11, we know that

$$\mathrm{H}_i^{\mathcal{S}}(S_4,\Gamma^{\varsigma}(\mathcal{O}_{\sigma_4},\mathcal{E})\otimes\xi_{\eta,\psi}^{-1})\cong\mathrm{H}_i^{\mathcal{S}}(\mathrm{GL}_2(\mathbb{R}), D_{k,\lambda}\widehat\otimes D_{k,z_0-\lambda}\otimes\eta_{\mathrm{GL}_2(\mathbb{R})}^{-1}),\quad i\in\mathbb{Z}.$$

Denote $\pi_0 := D_{k,\lambda}\widehat\otimes D_{k,z_0-\lambda}$. Using Borel's lemma, Shapiro's lemma, and Proposition 3.5, in order to prove the theorem, it suffices to prove that

$$\mathrm{H}_i^{\mathcal{S}}(P^\omega\cap S_4,(\delta_P^{\frac12}\pi_0)^\omega\otimes\mathrm{Sym}^l(\mathcal{N}_{\omega,\mathbb{C}}^*)\otimes\delta_{P^\omega\cap S_4}^{-1}\otimes\xi_{\eta,\psi}^{-1})=0,\quad\forall\ i,\ l\in\mathbb{Z}_{\geq 0},\tag{6.1}$$

where $\omega\in\{\sigma_1,\sigma_2,\sigma_3\}$. Since $\mathcal{O}_{\sigma_1}$ and $\mathcal{O}_{\sigma_3}$ are ψ-vanishing, (6.1) in these two cases follows from Lemma 5.1.

Since σ_2 is matching, the terms of modular characters can be canceled. Thus it suffices to prove

$$\mathrm{H}_i^{\mathcal{S}}(P^{\sigma_2}\cap S_4,\pi_0^{\sigma_2}\otimes\mathrm{Sym}^l(\mathcal{N}_{\sigma_2,\mathbb{C}}^*)\otimes\xi_{\eta,\psi}^{-1})=0,\quad\forall\ i,\ l\in\mathbb{Z}_{\geq 0}.$$

We first prove the case when $i=l=0$. One can easily calculate that $R_{\sigma_2}\cong S_2\times S_2$ and $V_{\sigma_2}=N^{\sigma_2}\cap S_4$, where N is the unipotent radical of P. Thus

$$\mathrm{H}_0^{\mathcal{S}}(P^{\sigma_2}\cap S_4,\pi_0^{\sigma_2}\otimes\xi_{\eta,\psi}^{-1})=\mathrm{H}_0^{\mathcal{S}}(S_2\times S_2,(D_{k,\lambda}\otimes\xi_{\eta,\psi}^{-1})\widehat\otimes(D_{k,z_0-\lambda}\otimes\xi_{\eta,\psi}^{-1}))=0,$$

where the second equality follows from Künneth formula and the condition that $D_{k,\lambda}$ doesn't admit a (η,ψ)-twisted Shalika peroids.

Now we consider the case when $i>0$ or $l>0$. A center element in $P^{\sigma_2}\cap S_4$ has form

$$\begin{bmatrix} a &&& \\ & d && \\ && a & \\ &&& d\end{bmatrix},$$

it acts on $\mathrm{H}_i^{\mathcal{S}}(V_{\sigma_2},\pi_0^{\sigma_2}\otimes\mathrm{Sym}^l(\mathcal{N}_{\sigma_2,\mathbb{C}}^*)\otimes\xi_{\eta,\psi}^{-1})$ by

$$|a|^{2\lambda-z_0+l+i}|d|^{z_0-2\lambda-l-i}(\operatorname{sgn}(ad))^{k+1+m_0+l+i}.$$

If $\mathrm{Re}(2\lambda - z_0) \geq 0$ then $2\lambda - z_0 + l + i \neq 0$. For the case $\mathrm{Re}(2\lambda - z_0) \leq 0$, [Moe97, Theorem 10b] asserts that $|2\lambda - z_0| \notin \{1, 2, 3, \dots\}$, thus $2\lambda - z_0 + l + i \neq 0$. Following Lem 3.11, we have

$$\mathrm{H}_j^{\mathcal{S}}(R_{\sigma_2}, \mathrm{H}_i^{\mathcal{S}}(V_{\sigma_2}, \pi_0^{\sigma_2} \otimes \mathrm{Sym}^k(\mathcal{N}^*_{\sigma_2,\mathbb{C}}) \otimes \xi_{\eta,\psi}^{-1})) = 0.$$

Thus following the spectral sequence argument,

$$\mathrm{H}_i^{\mathcal{S}}(P^{\sigma_2} \cap S_4, \pi_0^{\sigma_2} \otimes \mathrm{Sym}^l(\mathcal{N}^*_{\sigma_2,\mathbb{C}}) \otimes \xi_{\eta,\psi}^{-1}) = 0, \quad \forall\, i,\ l \in \mathbb{Z}_{\geq 0}.$$

□

Remark. The above proof also applies to the case of the limit of relative discrete series.

Proof of Theorem 1.2B. We first consider the case of $\mathrm{GL}_2(\mathbb{K})$. An irreducible representation π of $\mathrm{GL}_2(\mathbb{K})$ has an (η, ψ)-Shalika period if and only if it has a Whittaker period and the central character of π is η. For $\mathbb{K} = \mathbb{C}$, the η-symplectic L-parameter must has form $\chi + \chi^\vee \cdot \eta$. Assuming it is generic, then the corresponding irreducible principal series representation has central character η. For $\mathbb{K} = \mathbb{R}$, there are two cases. In the irreducible principal series case, it coincides with that for $\mathrm{GL}_2(\mathbb{C})$. For the irreducible 2-dimensional representation $\sigma_{k,\lambda}$ of $W_{\mathbb{R}}$, note that $\mathrm{GL}_2(\mathbb{C}) = \mathrm{GSp}_2(\mathbb{C})$, thus the similitude character of $\sigma_{k,\lambda}$ is $\det(\sigma_{k,\lambda}) = \mathrm{sgn}^{k+1} |\cdot|^{2\lambda}$, which is just the central character of $D_{k,\lambda}$. Here, we view $\det(\sigma_{k,\lambda})$ as a character of $\mathbb{R}^\times$ using reciprocity map.

We now turn to the general case of $\mathrm{GL}_{2n}(\mathbb{K})$. For $\mathbb{K} = \mathbb{C}$, the L-parameter of η-symplectic type has form

$$\sum_i (\chi_i + \chi_i^\vee \cdot \eta),$$

thus Theorem 1.2B follows from the case of $\mathrm{GL}_2(\mathbb{C})$ and Theorem 6.1.

For $\mathbb{K} = \mathbb{R}$, the L-parameter of η-symplectic type has form

$$\sum_i \sigma_{k_i,\lambda_i} + \sum_j (\phi_j + \phi_j^\vee \cdot \eta),$$

thus Theorem 1.2B follows from the case of $\mathrm{GL}_2(\mathbb{R})$, Theorem 6.2 and Theorem 6.1. □

7. Theta correspondence and linear periods

As proved in [Gan19], Shalika periods are related to linear periods under theta correspondence of the dual pair $(\mathrm{GL}_{2n}(\mathbb{K}), \mathrm{GL}_{2n}(\mathbb{K}))$. Denote by $\Theta(\pi)$ the full theta lift of a representation π.

Theorem 7.1 ([Gan19, Theorem 3.1])**.** *Let $\mathbb{K}$ be a non-archimedean local field of characteristic not 2. For any $\pi \in \mathrm{Irr}(\mathrm{GL}(W))$ and $\sigma \in \mathrm{Irr}(G_B)$, one has*

$$\mathrm{Hom}_{G_B \ltimes N(V_1)}(\Theta(\pi), \sigma \boxtimes \psi_B) \cong \mathrm{Hom}_{\mathrm{GL}(W_1)\times \mathrm{GL}(W_2)}(\pi^\vee, \sigma \boxtimes \mathbb{C})$$

where we have regarded σ naturally as a representation of $\mathrm{GL}(W_1)$. (All the notations here will be explained in the following).

In this section, we prove Theorem 7.5, which is an analogous result over the archimedean local field. The following Lemma 7.2 is essential to our proof.

Lemma 7.2 ([AGS15, Lemma 6.2.2])**.** *Let X be a Nash manifold and let V be a real vector space. Let $\phi : X \to V^*$ be a Nash map. Suppose $0 \in V^*$ is a regular value of ϕ. It gives a map $\chi : V \to \mathcal{T}(X)$ given by $\chi(v)(x) = \theta(\phi(x)(v))$ (where $\mathcal{T}(X)$ denotes the space of tempered functions on X). This gives an action of V on $S(X)$ by $\pi(v)(f) := \chi(v) \cdot f$. Then:*

(i) $\mathrm{H}_i(\mathfrak{v}, S(X)) = 0$ *for* $i > 0$.
(ii) Let $X_0 := \phi^{-1}(0)$. *Note that it is smooth. Let* r *denote the restriction map* $r : S(X) \to S(X_0)$. *Then* r *gives an isomorphism* $\mathrm{H}_0(\mathfrak{v}, S(X)) \xrightarrow{\sim} S(X_0)$.

We introduce the following notations for the Schrödinger model as in [Gan19, Section 3]. Let $\mathbb{K}$ be an archimedean local field. Let V and W be $2n$-dimensional vector spaces over $\mathbb{K}$. Write

$$V = V_1 + V_2 \text{ with } \dim V_i = n.$$

Denote by $P(V_1) = (\mathrm{GL}(V_1) \times \mathrm{GL}(V_2)) \ltimes N(V_1)$ the parabolic subgroup of $\mathrm{GL}(V)$ stabilizing V_1, where $N(V_1) \cong \mathrm{Hom}(V_2, V_1)$. For $A \in \mathrm{Hom}(V_2, V_1)$, we write $n(A)$ for the corresponding element in $N(V_1)$. We work on the Schrödinger model of the Weil representation

$$\Omega \cong \mathcal{S}((W \otimes V_1^*) \times (W \otimes V_2^*)).$$

The action of $P(V_1) \times \mathrm{GL}(W)$ is given by

$$\begin{cases} (h \cdot f)(T, X) = f(h^{-1} \circ T, X \circ h) & \text{for } h \in \mathrm{GL}(W); \\ ((g_1, g_2) \cdot f)(T, X) = (\det(g_1)/\det(g_2))^n \cdot f(T \circ g_1, g_2^{-1} \circ X) & \\ & \text{for } (g_1, g_2) \in \mathrm{GL}(V_1) \times \mathrm{GL}(V_2); \\ (n(A) \cdot f)(T, X) = \psi(\mathrm{Tr}_{V_2}(XTA)) \cdot f(T, X) & \text{for } A \in \mathrm{Hom}(V_2, V_1). \end{cases}$$

Given $B \in \mathrm{Isom}(V_1, V_2)$, define the character of $N(V_1)$ by

$$\psi_B : n(A) \mapsto \psi(\mathrm{Tr}_{V_2}(BA)).$$

The stabilizer of ψ_B in $\mathrm{GL}(V_1) \times \mathrm{GL}(V_2)$ is the diagonally embedded subgroup

$$G_B := \{(g, BgB^{-1}) |\ g \in \mathrm{GL}(V_1)\} \subset \mathrm{GL}(V_1) \times \mathrm{GL}(V_2).$$

The following lemma describes the coinvariance space of the Weil representation with respect to $(N(V_1), \psi_B)$.

Lemma 7.3.

$$\mathrm{H}_i^{\mathcal{S}}(N(V_1), \Omega \otimes \psi_B^{-1}) = \begin{cases} S(\mathcal{O}_B), & \textit{if } i = 0; \\ 0, & \textit{otherwise}, \end{cases}$$

where

$$\mathcal{O}_B := \{(T, X) \in \mathrm{Hom}(V_1, W) \times \mathrm{Hom}(W, V_2) |\ XT = B\}.$$

Proof. Note that $N(V_1)$ is an abelian group. Denote by $\mathfrak{n}(V_1)$ the Lie algebra of $N(V_1)$. Consider the following map

$$\begin{array}{cccc} F: & \mathrm{Hom}(V_1, W) \times \mathrm{Hom}(W, V_2) & \to & \mathfrak{n}^*(V_1) \\ & (T, X) & \mapsto & (A \mapsto \mathrm{Tr}((XT - B)A)). \end{array}$$

Note that $F^{-1}(0) = \mathcal{O}_B$, the lemma follows from Lemma 7.2 once we verify that 0 is a regular value of F. Take $(T_0, X_0) \in \mathcal{O}_B$, then

$$\frac{\partial F}{\partial(T_1, X_1)}(T_0, X_0) = (A \mapsto \mathrm{Tr}((X_1 T_0 + x_0 T_1)A)).$$

We may fix a basis of the above vector space so that the linear map can be expressed by matrices. Since T_0 is a $n \times 2n$ matrix with rank n, there exist $X_{i,j} \in \mathrm{Hom}(W, V_2)$ such that $X_{i,j} T_0 = E_{i,j}$, $\forall\ 1 \le i, j \le n$, where $E_{i,j}$ is the elementary matrix of size $n \times n$. Then

$$\frac{\partial F}{\partial(0, X_{i,j})}(T_0, X_0) = (A \mapsto a_{j,i}).$$

Thus 0 is a regular value of F. $\square$

Note that $G_B \times \mathrm{GL}(W)$ acts transitively on $\mathcal{O}_B$. Take $(T_0, X_0) \in \mathcal{O}_B$, so that

$$W = \mathrm{Im}(T_0) \oplus \mathrm{Ker}(X_0) =: W_1 \oplus W_2,$$

Then the stabilizer of (T_0, X_0) in $G_B \times \mathrm{GL}(W)$ is

$$H_B = \{((g, Bgb^{-1}), (T_0 g T_0^{-1}|_{W_1}, h))|\ g \in \mathrm{GL}(V_1),\ h \in \mathrm{GL}(W_2)\}.$$

As another ingredient of the proof of Theorem 7.5, the full theta lift of generic representation is studied in [FSX18].

Lemma 7.4 ([FSX18, Theorem 1.7])**.** *Let $\mathbb{K}$ be a local field. Assume σ is an irreducible generic representation of $\mathrm{GL}_n(\mathbb{K})$, then $\Theta(\sigma) \cong \sigma^\vee$.*

Now we are ready to prove the main theorem in this section.

Theorem 7.5. *The notation is as shown above. Let $\mathbb{K}$ be an archimedean local field. For any $\pi \in \mathrm{Irr}(\mathrm{GL}(W))$ and χ a character of G_B, one has*

$$\mathrm{Hom}_{G_B \ltimes N(V_1)}(\Theta(\pi), \chi \boxtimes \psi_B) \cong \mathrm{Hom}_{\mathrm{GL}(W_1)\times \mathrm{GL}(W_2)}(\pi^\vee, \chi \boxtimes \mathbb{C})$$

where we regard χ naturally as a character of $\mathrm{GL}(W_1)$. Moreover, if π is a generic representation, we have

$$\mathrm{Hom}_{G_B \ltimes N(V_1)}(\pi, \chi \boxtimes \psi) \cong \mathrm{Hom}_{\mathrm{GL}(W_1)\times \mathrm{GL}(W_2)}(\pi, \chi \boxtimes \mathbb{C})$$

Proof. The second assertion follows from the first one and Lemma 7.4. As to the first equality, the proof is similar to the original one in [Gan19], We sketch the proof for the convenience of the readers. Note that

$$\begin{aligned}
&\mathrm{H}_0^{\mathcal{S}}(G_B \ltimes N(V_1), \Theta(\pi) \otimes (\chi^{-1} \boxtimes \psi_B^{-1})) \\
=&\mathrm{H}_0^{\mathcal{S}}(G_B \ltimes N(V_1), \mathrm{H}_0^{\mathcal{S}}(\mathrm{GL}(W), \Omega \mathbin{\widehat{\otimes}} \pi^\vee) \otimes (\chi^{-1} \boxtimes \psi_B^{-1})) \\
=&\mathrm{H}_0^{\mathcal{S}}(G_B \times \mathrm{GL}(W), \mathrm{H}_0^{\mathcal{S}}(N(V_1), \Omega \otimes \psi_B^{-1}) \otimes \chi^{-1} \mathbin{\widehat{\otimes}} \pi^\vee) \\
=&\mathrm{H}_0^{\mathcal{S}}(G_B \times \mathrm{GL}(W), (\mathrm{ind}_{H_B}^{G_B \times \mathrm{GL}(W)} \mathbb{C}) \ \otimes \chi^{-1} \mathbin{\widehat{\otimes}} \pi^\vee) \\
=&\mathrm{H}_0^{\mathcal{S}}(\mathrm{GL}(W_1) \times \mathrm{GL}(W_2), \pi^\vee \otimes (\chi^{-1} \boxtimes \mathbb{C}))
\end{aligned}$$

where the third equality follows from Lemma 7.3 and the discussion after it, and the fourth equality follows from Shapiro's lemma. The first term occurring in the equality is finite-dimensional according to Corollary 5.11, thus the theorem follows by taking continuous dual. ☐

8. Restriction to $\mathrm{GL}_{2n}^+(\mathbb{R})$

In this section, we prove Theorem 1.6. We fix the following notations in this section. For $m \in \mathbb{Z}_{\geq 1}$, let $G_m := \mathrm{GL}_m(\mathbb{R})$ be the real general linear group of rank m, and let $G_m^+ := \mathrm{GL}_m^+(\mathbb{R})$ be the identity component of G_m. For $a \in \mathbb{R}^\times$, let

$$\psi_a : \mathbb{R} \to \mathbb{C}^\times,\ x \mapsto \exp(2\pi a x\sqrt{-1}).$$

Let $K_{2n} := \mathrm{O}_{2n}(\mathbb{R}) \supset K_{2n}^+ := \mathrm{SO}_{2n}(\mathbb{R})$. We first review the classification of the irreducible representations of K_{2n} and K_{2n}^+.

Fix the identification

$$\begin{array}{ccc}
\mathrm{SO}_2(\mathbb{R}) & \xrightarrow{\sim} & \mathrm{U}(1) \\
\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} & \mapsto & e^{i\theta}.
\end{array}$$

According to the highest weight theory, the irreducible representations of $\mathrm{SO}_{2n}(\mathbb{R})$ are parametrized by $(a_1,\ldots,a_n)\in\mathbb{Z}^n$ such that

$$a_1\geq\cdots\geq a_{n-1}\geq|a_n|.$$

Let σ be the irreducible representation of $\mathrm{SO}_{2n}(\mathbb{R})$ with highest weight $(a_1,\ldots,a_n)$. When $a_n=0$, there are exactly two non-isomorphic irreducible representations τ_1,τ_2 of $\mathrm{O}_{2n}(\mathbb{R})$, such that

$$\tau_i|_{\mathrm{O}_{2n}(\mathbb{R})}\cong\sigma,\quad i=1,\ 2.$$

When $a_n>0$, denote by σ' the representation corresponding to $(a_1,\ldots,-a_n)$, then there is a unique irreducible representation τ of $\mathrm{O}_{2n}(\mathbb{R})$, such that $\tau|_{\mathrm{O}_{2n}(\mathbb{R})}\cong\sigma\oplus\sigma'$. These give a parametrization of the representation of $\mathrm{O}_{2n}(\mathbb{R})$. When the second case happens, we denote $\tau^+:=\sigma$ and $\tau^-:=\sigma'$, thus

$$\tau|_{\mathrm{O}_{2n}(\mathbb{R})}\cong\tau^+\oplus\tau^-.$$

Let π be an irreducible representation of G_{2n}. According to Clifford theory, $\pi|_{G_{2n}^+}$ is either an irreducible representation of G_{2n}^+, or it is reducible and isomorphic to a direct sum of two non-isomorphic irreducible representations of G_{2n}^+. The following lemma describes the condition for the second case.

Lemma 8.1. *The notation is as shown in Subsection 2.1. Let π be an irreducible representation of G_{2n} with its L-parameter*

$$\phi_\pi=\sum_i\chi_{0,t_i}+\sum_j\chi_{1,s_j}+\sum_k\sigma_{l,r_k},\quad t_i,s_j,r_k\in\mathbb{C},$$

then $\pi|_{G_{2n}^+}$ is reducible if and only if $\chi_{0,t}$ and $\chi_{1,t}$ appear in pairs.

Proof. Following the Clifford theory, $\pi|_{G_{2n}^+}$ is reducible if and only if $\pi\cong\pi\otimes\mathrm{sgn}$. Then the lemma follows from the local Langlands correspondence. □

Denote the limit of relative discrete series of $\mathrm{GL}_2(\mathbb{R})$ by

$$D_{0,\lambda}:=|\cdot|^\lambda\dot{\times}\,\mathrm{sgn}\,|\cdot|^\lambda,\ \lambda\in\mathbb{C}.$$

Using induction by stage, the above lemma says that, when $\pi|_{G_{2n}^+}$ is reducible, the corresponding standard module has form

$$D_{k_1,\lambda_1}\dot{\times}\cdots\dot{\times}D_{k_n,\lambda_n},\ k_i\in\mathbb{Z}_{\geq0},\ \lambda_i\in\mathbb{C}.$$

Note that the minimal K_{2n}-type of this standard module (thus the irreducible quotient) has extremal weight $(k_1{+}1,\ldots,k_n{+}1)$. Denote by τ the minimal K_{2n}-type of π. Following the classification of the representations of K_{2n}, we have $\tau|_{\mathrm{SO}_{2n}}=\tau^+\oplus\tau^-$. We shall denote

$$\pi|_{G_{2n}^+}=\pi^+\oplus\pi^-$$

such that $\tau^+\subset\pi^+$, $\tau^-\subset\pi^-$. Note that π^+ and π^- can be gotten from each other by twisting a matrix g with $\det g=-1$.

Denote by S_{2n} the Shalika subgroup of G_{2n}. Note that $S_{2n}\subset G_{2n}^+$. For an irreducible representation π of G_{2n} such that $\pi|_{G_{2n}^+}=\pi^+\oplus\pi^-$, we have the following equation,

$$\mathrm{Hom}_{S_{2n}}(\pi,\xi_{\eta,\psi_a})=\mathrm{Hom}_{S_{2n}}(\pi^+,\xi_{\eta,\psi_a})\oplus\mathrm{Hom}_{S_{2n}}(\pi^-,\xi_{\eta,\psi_a}),$$

where the left hand side of the equation is at most one-dimensional. Assume further that π has a non-zero (η,ψ_a)-twisted Shalika period μ, we define

$$\epsilon_\pi:=\begin{cases}1, & \text{if } \mu|_{\pi^+}\neq0;\\ -1, & \text{if } \mu|_{\pi^-}\neq0.\end{cases}\tag{8.1}$$

In order to calculate ϵ_π, we first study the behavior of the parabolic induced representations of G_{2n} when restricted to G_{2n}^+. Let P be a standard parabolic subgroup of $\mathrm{GL}_{2n}(\mathbb{R})$ corresponding to the even partition $(n_1,\ldots,n_r)$ (i.e. all n_i are even), with Levi decomposition $P = L \ltimes N_P$. Let $P^+ := P \cap \mathrm{GL}_{2n}^+(\mathbb{R})$, $L^+ := L \cap \mathrm{GL}_{2n}^+(\mathbb{R})$, and let P° (resp. L°) be the identity component of P (resp. L), then we have $P^+ = L^+ \ltimes N_P$, $P^\circ = L^\circ \ltimes N_P$. Note that $L^\circ = G_{n_1}^+ \times \cdots \times G_{n_r}^+$.

Lemma 8.2. *Let P be as above, and let π_i be a irreducible representation of G_{n_i} such that $\pi_i|_{G_{n_i}^+} = \pi_i^+ \oplus \pi_i^-$, then we have*

$$(\mathrm{Ind}_P^{G_{2n}}\pi_1\widehat{\otimes}\cdots\widehat{\otimes}\pi_r)|_{G_{2n}^+} \cong (\mathrm{Ind}_{P^\circ}^{G_{2n}^+}\pi_1^+\widehat{\otimes}\cdots\widehat{\otimes}\pi_{r-1}^+\widehat{\otimes}\pi_r^+) \oplus (\mathrm{Ind}_{P^\circ}^{G_{2n}^+}\pi_1^+\widehat{\otimes}\cdots\widehat{\otimes}\pi_{r-1}^+\widehat{\otimes}\pi_r^-).$$

Proof. Note that $\pi_i \cong \mathrm{Ind}_{G_{n_i}^+}^{G_{n_i}}\pi^+$, thus $\pi_1\widehat{\otimes}\cdots\widehat{\otimes}\pi_r \cong \mathrm{Ind}_{L^\circ}^{L}\pi_1^+\widehat{\otimes}\cdots\widehat{\otimes}\pi_{r-1}^+\widehat{\otimes}\pi_r^+$. Denote $g' := \begin{bmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & & -1 \end{bmatrix} \in L \subset G_{2n}$, then $L^\circ\backslash L/L^+ = \{1, g'\}$. Thus

$$(\pi_1\widehat{\otimes}\cdots\widehat{\otimes}\pi_r)|_{L^+} \cong (\mathrm{Ind}_{L^\circ}^{L^+}\pi_1^+\widehat{\otimes}\cdots\widehat{\otimes}\pi_{r-1}^+\widehat{\otimes}\pi_r^+) \oplus (\mathrm{Ind}_{L^\circ}^{L^+}\pi_1^+\widehat{\otimes}\cdots\widehat{\otimes}\pi_{r-1}^+\widehat{\otimes}\pi_r^-).$$

Since $P\backslash G_{2n}/G_{2n}^+ = \{1\}$, we have

$$\begin{aligned}\pi|_{G_{2n}^+} \cong& \mathrm{Ind}_{P^+}^{G_{2n}^+}(\pi_1\widehat{\otimes}\cdots\widehat{\otimes}\pi_r|_{L^+})\\ \cong&(\mathrm{Ind}_{P^\circ}^{G_{2n}^+}\pi_1^+\widehat{\otimes}\cdots\widehat{\otimes}\pi_{r-1}^+\widehat{\otimes}\pi_r^+) \oplus (\mathrm{Ind}_{P^\circ}^{G_{2n}^+}\pi_1^+\widehat{\otimes}\cdots\widehat{\otimes}\pi_{r-1}^+\widehat{\otimes}\pi_r^-).\end{aligned}$$

□

For $\mathrm{GL}_2(\mathbb{R})$, as we have mentioned below Theorem 6.1, the Shalika period is closely related to the Whittaker period, which has been thoroughly studied in [God18]. We have the following lemma.

Lemma 8.3. *Let $\pi := D_{k,\lambda}$, $k \in \mathbb{Z}_{\geq 0}$, $\lambda \in \mathbb{C}$, then $\epsilon_\pi = \operatorname{sgn} a$.*

Proof. The case when $a > 0$ has been calculated in [God18, Section 2.5] and [CC19, Section 3.1]. Then the case when $a < 0$ can be obtained by twisting $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$. □

For $\mathrm{GL}_4(\mathbb{R})$, we have the following lemma, which is similar to Theorem 6.2.

Lemma 8.4. *Let $P \subset \mathrm{GL}_4(\mathbb{R})$ be the standard parabolic subgroup corresponding to the partition $(2,2)$. Denote by $\eta : \mathbb{R}^\times \to \mathbb{C}^\times$, $t \mapsto |t|^{z_0}(\operatorname{sgn} t)^{m_0}$ a character of $\mathbb{R}^\times$. Assume $D_{k,\lambda}$ ($k \in \mathbb{Z}_{\geq 0}$, $\lambda \in \mathbb{C}$) doesn't admit a η-twisted Shalika periods (thus so is $D_{k,z_0-\lambda}$), and $\pi := D_{k,\lambda}\dot{\times}D_{k,z_0-\lambda}$ is irreducible, then*

$$D_{k,\lambda}^+\dot{\times}D_{k,z_0-\lambda}^- := \mathrm{Ind}_{P^\circ}^{\mathrm{GL}_4^+(\mathbb{R})}D_{k,\lambda}^+\widehat{\otimes}D_{k,z_0-\lambda}^-$$

admits a (η,ψ)-twisted Shalika peroids. Moreover, we have

$$\mathrm{H}_i^{\mathcal{S}}(S_4, D_{k,\lambda}^+\dot{\times}D_{k,z_0-\lambda}^- \otimes \xi_{\eta,\psi}^{-1}) \cong \mathrm{H}_i^{\mathcal{S}}(\mathrm{GL}_2^+(\mathbb{R}), D_{k,\lambda}^+\widehat{\otimes}D_{k,z_0-\lambda}^- \otimes \eta_{\mathrm{GL}_2^+(\mathbb{R})}^{-1}), \quad i \in \mathbb{Z},$$

and

$$\mathrm{Hom}_{S_4}(D_{k,\lambda}^+\dot{\times}D_{k,z_0-\lambda}^-, \xi_{\eta,\psi}) \cong \mathrm{Hom}_{\mathrm{GL}_2^+(\mathbb{R})}(D_{k,\lambda}^+\widehat{\otimes}D_{k,z_0-\lambda}^-, \eta_{\mathrm{GL}_2^+(\mathbb{R})}).$$

Thus $\epsilon_\pi = -1$.

Proof. The proof is similar to the one of Theorem 6.2, we only sketch it here for the convenience of the readers. Let

$$g' := \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & -1 \end{bmatrix}, g_0 := \begin{bmatrix} 1 & & & \\ & -1 & & \\ & & 1 & \\ & & & -1 \end{bmatrix}.$$

Recall that in Theorem 6.2, we have the following representative $\{\sigma_1 = (1,2,3,4), \sigma_2 = (1,3,2,4), \sigma_3 = (3,1,2,4), \sigma_4 = (3,4,1,2)\}$. In this case, we have

$$P^\circ \backslash G_{2n}^+ / S_4 = \{\sigma_1, g'\sigma_2, g_0 g'\sigma_2, \sigma_3, g_0\sigma_3, \sigma_4\}.$$

One can use the same argument as in Theorem 6.2 to show that

$$\mathrm{H}_i^{\mathcal{S}}(S_4, D_{k,\lambda}^+ \dot{\times} D_{k,z_0-\lambda}^- \otimes \xi_{\eta,\psi}^{-1}) \cong \mathrm{H}_i^{\mathcal{S}}(\mathrm{GL}_2^+(\mathbb{R}), D_{k,\lambda}^+ \widehat{\otimes} D_{k,z_0-\lambda}^- \otimes \eta_{\mathrm{GL}_2^+(\mathbb{R})}^{-1}), \quad i \in \mathbb{Z},$$

where the right term has dimension one for $i = 0$. Thus

$$\dim \mathrm{Hom}_{S_4}(D_{k,\lambda}^+ \dot{\times} D_{k,z_0-\lambda}^-, \xi_{\eta,\psi}) = 1,$$

and $\epsilon_\pi = -1$ □

The following lemma is an analog of Theorem 6.1 for G_{2n}^+.

Lemma 8.5. *For two even positive integers n_1 and n_2, take two Casselman-Wallach representations π_1 and π_2 of $\mathrm{GL}_{n_1}^+(\mathbb{R})$ and $\mathrm{GL}_{n_2}^+(\mathbb{R})$, respectively. Then the normalized parabolic induction $\pi := \mathrm{Ind}_{P_{n_1,n_2}^\circ}^{\mathrm{GL}_{n_1+n_2}^+(\mathbb{R})} \pi_1 \widehat{\otimes} \pi_2$ has a non-zero (η,ψ)-Shalika periods if one of the following condition holds:*

(1) $n_1 \equiv 0$ or $n_2 \equiv 0 \mod 4$, and both π_1 and π_2 have (η,ψ)-Shalika periods.

(2) $n_1 \equiv 2$ and $n_2 \equiv 2 \mod 4$, π_1 has (η,ψ^{-1})-Shalika period and π_2 has (η,ψ)-Shalika period.

Proof. Let $n_i = 2m_i$, $\omega_s' := \begin{bmatrix} 1_{m_1} & & & \\ & 0 & 1_{m_2} & \\ & 1_{m_1} & 0 & \\ & & & 1_{m_2} \end{bmatrix}$, then $\det \omega_s' = (-1)^{m_1 m_2}$. Thus for the first case, $\omega_s' \in G_{2n}^+$ and the proof is exactly the same as the original one in [CJLT20, Theorem 2.1]. We only treat the remaining case and point out the difference. For the second case, take $\omega_s := \begin{bmatrix} -1_{m_1} & & & \\ & 0 & 1_{m_2} & \\ & 1_{m_1} & 0 & \\ & & & 1_{m_2} \end{bmatrix} \in G_{2n}^+$. Then we have

$$\omega_s^{-1} \begin{bmatrix} 1_{m_1} & 0 & X & Z \\ & 1_{m_2} & 0 & Y \\ & & 1_{m_1} & 0 \\ & & & 1_{m_2} \end{bmatrix} \omega_s = \begin{bmatrix} 1_{m_1} & -X & 0 & -Z \\ & 1_{m_2} & 0 & 0 \\ & & 1_{m_1} & Y \\ & & & 1_{m_2} \end{bmatrix},$$

and

$$\omega_s^{-1} \begin{bmatrix} 1_{m_1} & X & 0 & 0 \\ & 1_{m_2} & 0 & 0 \\ & & 1_{m_1} & Y \\ & & & 1_{m_2} \end{bmatrix} \omega_s = \begin{bmatrix} 1_{m_1} & 0 & -X & 0 \\ & 1_{m_2} & 0 & Y \\ & & 1_{m_1} & 0 \\ & & & 1_{m_2} \end{bmatrix}.$$

For $f \in \mathrm{Ind}_{P_{n_1,n_2}^\circ}^{\mathrm{GL}_{n_1+n_2}^+(\mathbb{R})} \pi_1 \widehat{\otimes} \pi_2$, we define a function on G_{2n}^+ by $\Phi(g; f) := < \mu_1 \otimes \mu_2, f(\omega_s^{-1} g) >$, where μ_i is the non-zero twisted Shalika functional on π_i. Then we obtain

$$\begin{aligned} \Phi\left(\begin{bmatrix} 1_{m_1} & 0 & X & Z \\ & 1_{m_2} & 0 & Y \\ & & 1_{m_1} & 0 \\ & & & 1_{m_2} \end{bmatrix} g; f \right) =& \psi^{-1}(-\mathrm{Tr}(X)) \cdot \psi(\mathrm{Tr}(Y)) \cdot \Phi(g; f), \\ =& \psi(\mathrm{Tr}(X) + \mathrm{Tr}(Y)) \cdot \Phi(g; f) \end{aligned}$$

and

$$\Phi\left(\begin{bmatrix} 1_{m_1} & X & 0 & 0 \\ & 1_{m_2} & 0 & 0 \\ & & 1_{m_1} & Y \\ & & & 1_{m_2} \end{bmatrix} g; f\right) = \Phi(g; f).$$

The subsequent construction of the twisted Shalika functional on π follows the same process as in [CJLT20, Theorem 2.1]. $\square$

We are now prepared to state the main theorem in this section.

Theorem 8.6. *Let π be an irreducible generic representation of G_{2n}. Assume π admits a non-zero (η, ψ_a)-twisted Shalika period and $\pi|_{G_{2n}^+}$ is reducible, then π has form*

$$D_{k_1,\lambda_1} \dot{\times} \cdots \dot{\times} D_{k_n,\lambda_n}, \; k_i \in \mathbb{Z}_{\geq 0}, \; \lambda_i \in \mathbb{C}.$$

Let $p := \#\{1 \leq i \leq n | D_{a_i,\lambda_i}$ has (η, ψ_a)-twisted Shalika period$\}$ and $q := \frac{n-p}{2}$, which are both integers since ϕ_π is of η-symplectic type. Then

$$\epsilon_\pi = (\mathrm{sgn} \;\; a)^p \cdot (-1)^{\frac{p(p-1)}{2}+q}.$$

Proof. Applying Lemma 8.3, 8.4, 8.5, the theorem is derived through an induction argument. $\square$

Appendix A. Künneth formula and spectral sequence for nilpotent normal subgroups

A.1. **Künneth formula.** In this subsection, we prove a Künneth formula for Schwartz homology. Since $\mathcal{S}\mathrm{mod}_G$ is not an Abelian category, we first prove the following two basic lemma about homological algebra in the topological setting.

Lemma A.1 (Topological snake lemma)**.** *Consider a commutative diagram in the category of (not necessarily Hausdorff) topological vector spaces*

$$\begin{tikzcd} & V_1 \arrow[r, "\alpha_1"] \arrow[d, "\phi_1"] & V_2 \arrow[r, "\alpha_2"] \arrow[d, "\phi_2"] & V_3 \arrow[r] \arrow[d, "\phi_3"] & 0 \\ 0 \arrow[r] & W_1 \arrow[r, "\beta_1"] & W_2 \arrow[r, "\beta_2"] & W_3 & \end{tikzcd}$$

where the rows are weak exact sequences. If α_2 is an open map, $\beta_1 : W_1 \xrightarrow{\sim} \mathrm{Im}\,\beta_1$ is a topological isomorphism, then we have a weak exact sequence

$$\mathrm{Ker}(\phi_1) \to \mathrm{Ker}(\phi_2) \to \mathrm{Ker}(\phi_3) \xrightarrow{\delta} \mathrm{coker}(\phi_1) \to \mathrm{coker}(\phi_2) \to \mathrm{coker}(\phi_3)$$

where $\mathrm{Ker}(\phi_i)$ are equipped with subspace topology of V_i, $\mathrm{coker}(\phi_i)$ are equipped with quotient topology of W_i.

Proof. The conditions of this lemma are slightly different from those of [Sch99, Proposition 4], but the proof is identical. We refer the readers to [Sch99] for details. $\square$

Lemma A.2. *Let*

$$0 \to A_\cdot \xrightarrow{\phi_\cdot} B_\cdot \xrightarrow{\eta_\cdot} C_\cdot \to 0$$

be a short exact sequence of chain complexes of Fréchet spaces, where each ϕ_i is a strict map. Then we have a weak exact sequence

$$\cdots \to \mathrm{H}_{i+1}(C_\cdot) \to \mathrm{H}_i(A_\cdot) \to \mathrm{H}_i(B_\cdot) \to \mathrm{H}_i(C_\cdot) \to \mathrm{H}_{i-1}(A_\cdot) \to \cdots$$

Proof. This follows the standard argument based on Lemma A.1. We omit the details. $\square$

We also need the following lemma since we work in the topological setting.

Lemma A.3 ([CS21, Lemma 8.6])**.** *Let $\phi : \mathcal{C}_\bullet \to \mathcal{C}'_\bullet$ be a morphism of chain complexes of Fréchet spaces. Let $i \in \mathbb{Z}$. If the induced morphism*

$$\phi : \mathrm{H}_i(\mathcal{C}_\bullet) \to \mathrm{H}_i(\mathcal{C}'_\bullet)$$

is surjective, then it must be an open map.

We first define the completed tensor product of two bounded below chain complexes of topological vector spaces. For simplicity, we shall assume that the complex starts from 0.

Definition A.4. *Let $X_., Y_.$ be two bounded below chain complexes of topological vector spaces, both starting from 0. We define the completed tensor product of $X_.$ and $Y_.$ by*

$$(X_.\widehat{\otimes}Y_.)_m = \bigoplus_{p+q=m} X_p\widehat{\otimes}Y_q, \quad \forall\ m \in \mathbb{Z}$$

with the direct sum topology. The chain maps are the same as the algebraic tensor product of chain complexes.

Theorem A.5 (Künneth formula for chain complex)**.** *Let $X_., Y_.$ be two bounded below chain complexes of NF-spaces, both starting from 0. Assume $\forall\ k \in \mathbb{Z}$, $\mathrm{H}_k(X_.)$, $\mathrm{H}_k(Y_.)$ are NF-spaces. Then we have*

$$\mathrm{H}_m(X_.\widehat{\otimes}Y_.) \cong \bigoplus_{p+q=m} \mathrm{H}_p(X_.)\widehat{\otimes}\,\mathrm{H}_q(Y_.), \quad \forall\ m \in \mathbb{Z},$$

as topological vector spaces. In particular, $\mathrm{H}_m(X_.\widehat{\otimes}Y_.)$ are NF-spaces.

Proof. The proof is similar to the classical case, except for certain topological issues (where we use the nuclear Fréchet condition).

We first consider the case that $Y_i = 0$ for $i \neq p$, for some $p \in \mathbb{Z}$. For $m \in \mathbb{Z}$, according to Lemma 2.4, we have short exact sequences of NF-spaces

$$0 \to (\mathrm{Ker}\ d_m)\widehat{\otimes}Y_p \to X_m\widehat{\otimes}Y_p \xrightarrow{d_m\widehat{\otimes}1_{Y_p}} (\mathrm{Im}\ d_m)\widehat{\otimes}Y_p \to 0,$$

$$0 \to (\mathrm{Im}\ d_{m+1})\widehat{\otimes}Y_p \to (\mathrm{Ker}\ d_m)\widehat{\otimes}Y_p \to \mathrm{H}_m(X_.)\widehat{\otimes}Y_p \to 0.$$

Then we know that $(\mathrm{Im}\ d_{m+1})\widehat{\otimes}Y_p = \mathrm{Im}\ (d_{m+1}\widehat{\otimes}1_{Y_p})$, $(\mathrm{Ker}\ d_m)\widehat{\otimes}Y_p = \mathrm{Ker}\ (d_m\widehat{\otimes}1_{Y_p})$. Thus $\mathrm{H}_{m+p}(X_.\widehat{\otimes}Y_.) = \mathrm{H}_m(X_.)\widehat{\otimes}Y_p = \mathrm{H}_m(X_.)\widehat{\otimes}\,\mathrm{H}_p(Y_.)$.

Secondly, we consider the case that the differential map $d^Y_.$ in $Y_.$ are all 0. Note that $(X_.\widehat{\otimes}Y_.)_m = \bigoplus_{p+q=m} X_p\widehat{\otimes}Y_q$. Since $d^Y_. = 0$, the differential map $(X_.\widehat{\otimes}Y_.)_{m+1} \to (X_.\widehat{\otimes}Y_.)_m$ is just direct sum of family of maps $d^X_{p+1}\widehat{\otimes}1_{Y_q} : X_{p+1}\widehat{\otimes}Y_q \to X_p\widehat{\otimes}Y_q$. Since all the occurring spaces are NF-spaces, we have $(\mathrm{Im}\ d^X_{p+1})\widehat{\otimes}Y_q = \mathrm{Im}\ (d^X_{p+1}\widehat{\otimes}1_{Y_q})$, $(\mathrm{Ker}\ d^X_p)\widehat{\otimes}Y_q = \mathrm{Ker}\ (d^X_p\widehat{\otimes}1_{Y_q})$. Thus we obtain

$$\mathrm{H}_m(X_.\widehat{\otimes}Y_.) = \frac{\bigoplus_{p+q=m}(\mathrm{Ker}\ d^X_p)\widehat{\otimes}Y_q}{\bigoplus_{p+q=m}(\mathrm{Im}\ d^X_{p+1})\widehat{\otimes}Y_q} = \bigoplus_{p+q=m} \mathrm{H}_p(X_.)\widehat{\otimes}Y_q = \bigoplus_{p+q=m} \mathrm{H}_p(X_.)\widehat{\otimes}\,\mathrm{H}_q(Y_.).$$

For the general case, denote $Z(X)_m := \mathrm{Ker}\ d^X_m$, $B(X)_m := \mathrm{Im}\ d^X_{m+1}$. Let $Z(X)_.$ and $B(X)_.$ be the corresponding chain complexes with zero differential maps. Define the chain complex $\Sigma B(X)$ by setting $(\Sigma B(X))_{n+1} := B(X)_n$. We have a short exact sequence of chain complexes

$$0 \to Z(X)_. \to X_. \xrightarrow{d^X_.} \Sigma B(X)_. \to 0.$$

According to the condition, all these spaces are NF-spaces, thus we have

$$0 \to Z(X)_{\cdot}\widehat{\otimes}Y_{\cdot} \to X_{\cdot}\widehat{\otimes}Y_{\cdot} \xrightarrow{d_{\cdot}^{X}} \Sigma B(X)_{\cdot}\widehat{\otimes}Y_{\cdot} \to 0,$$

with the first maps being strict. Following Lemma A.2, we obtain

$$\begin{aligned}\cdots \to \mathrm{H}_{m+1}(\Sigma B(X)_{\cdot}\widehat{\otimes}Y_{\cdot}) \to \mathrm{H}_m(Z(X)_{\cdot}\widehat{\otimes}Y_{\cdot}) \to \mathrm{H}_m(X_{\cdot}\widehat{\otimes}Y_{\cdot})& \\ \to \mathrm{H}_m(\Sigma B(X)_{\cdot}\widehat{\otimes}Y_{\cdot}) \to \mathrm{H}_{m-1}(Z(X)_{\cdot}\widehat{\otimes}Y_{\cdot}) \to \cdots .&\end{aligned}$$

The first two terms, as well as the last two, can be computed using the second case, then we get the exact sequence

$$\begin{aligned}\cdots \to \bigoplus_{p+q=m} B(X)_p\widehat{\otimes}\,\mathrm{H}_q(Y_{\cdot}) \to \bigoplus_{p+q=m} Z(X)_p\widehat{\otimes}\,\mathrm{H}_q(Y_{\cdot}) \to \mathrm{H}_m(X_{\cdot}\widehat{\otimes}Y_{\cdot})& \\ \to \bigoplus_{p+q=m-1} B(X)_p\widehat{\otimes}\,\mathrm{H}_q(Y_{\cdot}) \to \bigoplus_{p+q=m-1} Z(X)_p\widehat{\otimes}\,\mathrm{H}_q(Y_{\cdot}) \to \cdots ,&\end{aligned}$$

where the connecting homomorphism is induced by the inclusion

$$B(X)_p\widehat{\otimes}\,\mathrm{H}_q(Y_{\cdot}) \hookrightarrow Z(X)_p\widehat{\otimes}\,\mathrm{H}_q(Y_{\cdot}).$$

Thus we have the short exact sequence

$$0 \to \bigoplus_{p+q=m} B(X)_p\widehat{\otimes}\,\mathrm{H}_q(Y_{\cdot}) \to \bigoplus_{p+q=m} Z(X)_p\widehat{\otimes}\,\mathrm{H}_q(Y_{\cdot}) \to \mathrm{H}_m(X_{\cdot}\widehat{\otimes}Y_{\cdot}) \to 0.$$

Following Lemma A.3, the surjective map above is an open map. Also, we know

$$0 \to B(X)_p\widehat{\otimes}\,\mathrm{H}_q(Y_{\cdot}) \to Z(X)_p\widehat{\otimes}\,\mathrm{H}_q(Y_{\cdot}) \to \mathrm{H}_p(X_{\cdot})\widehat{\otimes}\,\mathrm{H}_q(Y_{\cdot}) \to 0,$$

thus we get

$$\mathrm{H}_m(X_{\cdot}\widehat{\otimes}Y_{\cdot}) \cong \bigoplus_{p+q=l} \mathrm{H}_p(X_{\cdot})\widehat{\otimes}\,\mathrm{H}_q(Y_{\cdot})$$

as topological vector spaces. □

Now we focus on the Schwartz homology.

Lemma A.6. *Let G be an almost linear Nash group. Assume that $V,\ W \in \mathcal{S}\mathrm{mod}_G$ are both NF-spaces. If $\forall\ m \in \mathbb{Z}$, $\mathrm{H}_m^{\mathcal{S}}(G,V)$ are NF-spaces, then*

$$\mathrm{H}_m^{\mathcal{S}}(G, V\widehat{\otimes}W) = \mathrm{H}_m^{\mathcal{S}}(G,V)\widehat{\otimes}W, \quad \forall m \in \mathbb{Z},$$

as topological vector spaces.

Proof. Take $P_{\cdot} \twoheadrightarrow V$ to be a strong relative projective resolution of V, with $P_{\cdot}$ all NF-spaces, then $P_{\cdot}\widehat{\otimes}W \twoheadrightarrow V\widehat{\otimes}W$ is a strong relative projective resolution of $V\widehat{\otimes}W$.

We first compare $\sum(g-1)(P_k\widehat{\otimes}W)$ and $(\sum(g-1)P_k)\widehat{\otimes}W$, which are both closed subspace in $P_k\widehat{\otimes}W$. Note that $\sum(g-1)(P_k\otimes W) = (\sum(g-1)P_k)\otimes W$, $\sum(g-1)(P_k\otimes W)$ is dense in $\sum(g-1)(P_k\widehat{\otimes}W)$, $(\sum(g-1)P_k)\otimes W$ is dense in $(\sum(g-1)P_k)\widehat{\otimes}W$. Thus we have $\sum(g-1)(P_k\widehat{\otimes}W) = (\sum(g-1)P_k)\widehat{\otimes}W$, and $(P_k\widehat{\otimes}W)_G = (P_k)_G\widehat{\otimes}W$.

Consider the following sequence of NF-spaces

$$\cdots \to (P_{m+1})_G \xrightarrow{d_{m+1}} (P_m)_G \xrightarrow{d_m} (P_{m-1})_G \to \cdots .$$

According to Lemma 2.4, we have $(\mathrm{Im}\ d_{m+1})\widehat{\otimes}W = \mathrm{Im}\ (d_{m+1}\widehat{\otimes}1_W)$, $(\mathrm{Ker}\ d_m)\widehat{\otimes}W = \mathrm{Ker}\ (d_m\widehat{\otimes}1_W)$. Note that

$$0 \to (\mathrm{Im}\ d_{m+1})\widehat{\otimes}W \to (\mathrm{Ker}\ d_m)\widehat{\otimes}W \to \mathrm{H}_m^{\mathcal{S}}(G,V)\widehat{\otimes}W \to 0,$$

thus $\mathrm{H}_m^{\mathcal{S}}(G, V\widehat{\otimes}W) = \mathrm{H}_m^{\mathcal{S}}(G,V)\widehat{\otimes}W, \quad \forall m \in \mathbb{Z}.$ □

Theorem A.7 (Künneth formula for Schwartz homology)**.** *Let G_1 and G_2 be two almost linear Nash groups. Assume $V_i \in \mathcal{S}\mathrm{mod}_{G_i}$, $i = 1, 2$, are both NF-spaces. If $\forall\ j \in \mathbb{Z}$, $\mathrm{H}^{\mathcal{S}}_j(G_i, V_i)$ are NF-spaces, then*

$$\mathrm{H}^{\mathcal{S}}_m(G_1 \times G_2, V_1 \widehat{\otimes} V_2) \cong \bigoplus_{p+q=m} \mathrm{H}^{\mathcal{S}}_p(G_1, V_1) \widehat{\otimes}\, \mathrm{H}^{\mathcal{S}}_q(G_2, V_2), \quad \forall\ m \in \mathbb{Z},$$

as topological vector spaces. In particular, $\mathrm{H}^{\mathcal{S}}_m(G_1 \times G_2, V_1 \widehat{\otimes} V_2)$ are NF-spaces.

Proof. Using Theorem A.5, it suffices to prove that the Koszul complex for computing the Schwartz homology of $V_1 \widehat{\otimes} V_2$ in $\mathcal{S}\mathrm{mod}_{G_1 \times G_2}$ is just the completed tensor product of the Koszul complexes for V_i in $\mathcal{S}\mathrm{mod}_{G_i}$ $(i = 1, 2)$. Denote by K_i $(i = 1, 2)$ the maximal compact subgroup of G_i $(i = 1, 2)$.

Since V_i $(i = 1, 2)$ are both NF-spaces and K_i $(i = 1, 2)$ are both compact, we have

$$(\wedge^n(\mathfrak{g}_1 \oplus \mathfrak{g}_2/\mathfrak{k}_1 \oplus \mathfrak{k}_2) \otimes V_1 \widehat{\otimes} V_2)_{K_1 \times K_2} = \bigoplus_{p+q=n} ((\wedge^p(\mathfrak{g}_1/\mathfrak{k}_1) \otimes V_1) \widehat{\otimes} (\wedge^q(\mathfrak{g}_2/\mathfrak{k}_2) \otimes V_2))_{K_1 \times K_2}.$$

Thus we only need to consider $((\wedge^p(\mathfrak{g}_1/\mathfrak{k}_1) \otimes V_1) \widehat{\otimes} (\wedge^q(\mathfrak{g}_2/\mathfrak{k}_2) \otimes V_2))_{K_1 \times K_2}$. Note that K_i $(i = 1, 2)$ are both compact, thus $\mathrm{H}^{\mathcal{S}}_0(K_i, \wedge^p(\mathfrak{g}_i/\mathfrak{k}_i) \otimes V_i)$ $(i = 1, 2)$ are both NF-spaces according to Proposition 3.6. Following Lemma A.6, we have

$$\mathrm{H}^{\mathcal{S}}_0(K_1 \times K_2, (\wedge^p(\mathfrak{g}_1/\mathfrak{k}_1) \otimes V_1) \widehat{\otimes} (\wedge^q(\mathfrak{g}_2/\mathfrak{k}_2) \otimes V_2)) = \mathrm{H}^{\mathcal{S}}_0(K_1, \wedge^p(\mathfrak{g}_1/\mathfrak{k}_1) \otimes V_1) \widehat{\otimes}\, \mathrm{H}^{\mathcal{S}}_0(K_2, \wedge^q(\mathfrak{g}_2/\mathfrak{k}_2) \otimes V_2).$$

□

A.2. **Hochschild-Serre spectral sequence.** In this subsection, we aim to prove a Hochschild-Serre spectral sequence for nilpotent normal subgroups. Firstly, we recall the following spectral sequences in the category $\mathcal{C}_{\mathfrak{g}_{\mathbb{C}}, K}$ of $(\mathfrak{g}_{\mathbb{C}}, K)$-modules.

Theorem A.8 ([BW80, Theorem 6.5], [KV16, Corollary 3.6])**.** *Let $\mathfrak{h}$ be an ideal of $\mathfrak{g}$ stable under K, $\mathfrak{l} = \mathfrak{k} \cap \mathfrak{h}$, L a closed normal subgroup of K with Lie algebra $\mathfrak{l}$, and $W \in \mathcal{C}_{\mathfrak{g}_{\mathbb{C}}, K}$. Then there exists convergent first quadrant spectral sequences:*

$$E^2_{p,q} = \mathrm{H}_p(\mathfrak{g}_{\mathbb{C}}/\mathfrak{h}_{\mathbb{C}}, K/L; H_q(\mathfrak{h}_{\mathbb{C}}, L; W)) \implies H_{p+q}(\mathfrak{g}_{\mathbb{C}}, K; W)$$

Since we work in $\mathcal{S}\mathrm{mod}_G$, it is desirable to obtain a similar spectral sequence applicable to this category. The following lemma plays a transitional role.

Lemma A.9. *Let $H = L \ltimes N$ be an almost linear Nash group with nilpotent normal subgroup N, denote by M the maximal compact subgroup of L (and hence also the maximal compact subgroup in H). Consider $V \in \mathcal{S}\mathrm{mod}_H$, for a fixed $j \in \mathbb{Z}$, if $\mathrm{H}^{\mathcal{S}}_j(N, V) \in \mathcal{S}\mathrm{mod}_L$, then $\mathrm{H}_j(\mathfrak{n}_{\mathbb{C}}, V^{M\text{-fin}}) = \mathrm{H}^{\mathcal{S}}_j(N, V)^{M\text{-fin}}$*

Proof. Consider the Koszul complex for $\mathrm{H}_j(\mathfrak{n}_{\mathbb{C}}, V)$:

$$\cdots \to \wedge^{j+1}\mathfrak{n}_{\mathbb{C}} \otimes V \xrightarrow{d_{j+1}} \wedge^j \mathfrak{n}_{\mathbb{C}} \otimes V \xrightarrow{d_j} \wedge^{j-1}\mathfrak{n}_{\mathbb{C}} \otimes V \to \cdots.$$

Since $\mathrm{H}_j(\mathfrak{n}_{\mathbb{C}}, V) = \mathrm{H}^{\mathcal{S}}_j(N, V) \in \mathcal{S}\mathrm{mod}_L$, we have the following short exact sequence in $\mathcal{S}\mathrm{mod}_L$

$$0 \to \mathrm{Im}\ d_{j+1} \to \mathrm{Ker}\ d_j \to \mathrm{H}_j(\mathfrak{n}_{\mathbb{C}}, V) \to 0.$$

Note that taking M-finite vectors $-^{M\text{-fin}}$ is an exact functor from $\mathcal{S}\mathrm{mod}_L$ to $(\mathfrak{l}_{\mathbb{C}}, M)$-modules, thus we have

$$0 \to (\mathrm{Im}\ d_{j+1})^{M\text{-fin}} \to (\mathrm{Ker}\ d_j)^{M\text{-fin}} \to \mathrm{H}_j(\mathfrak{n}_{\mathbb{C}}, V)^{M\text{-fin}} \to 0.$$

We also have the following Koszul complex for $\mathrm{H}_j(\mathfrak{n}_{\mathbb{C}}, V^{M\text{-fin}})$:

$$\cdots \to \wedge^{j+1}\mathfrak{n}_{\mathbb{C}} \otimes V^{M\text{-fin}} \xrightarrow{d_{j+1}|_{M\text{-fin}}} \wedge^{j}\mathfrak{n}_{\mathbb{C}} \otimes V^{M\text{-fin}} \xrightarrow{d_{j}|_{M\text{-fin}}} \wedge^{j-1}\mathfrak{n}_{\mathbb{C}} \otimes V^{M\text{-fin}} \to \cdots .$$

Then we obtain

$$0 \to \mathrm{Im}\ (d_{j+1}|_{M\text{-fin}}) \to \mathrm{Ker}\ (d_j|_{M\text{-fin}}) \to \mathrm{H}_j(\mathfrak{n}_{\mathbb{C}}, V^{M\text{-fin}}) \to 0.$$

Thus it suffices to prove

$$\mathrm{Im}(d_{j+1}|_{M\text{-fin}}) = (\mathrm{Im} d_{j+1})^{M\text{-fin}},\ \mathrm{Ker}(d_j|_{M\text{-fin}}) = (\mathrm{Ker} d_j)^{M\text{-fin}}.$$

The first equality follows from the following short exact sequence

$$0 \to (\mathrm{Ker}\ d_{j+1})^{M\text{-fin}} \to (\wedge^{j+1}\mathfrak{n}_{\mathbb{C}} \otimes V)^{M\text{-fin}} \xrightarrow{d_{j+1}|_{M\text{-fin}}} (\mathrm{Im}\ d_{j+1})^{M\text{-fin}} \to 0,$$

and the second follows from $\mathrm{Ker}\ (d_j|_{M\text{-fin}}) = (\mathrm{Ker}\ d_j) \cap (\wedge^j \mathfrak{n}_{\mathbb{C}} \otimes V^{M\text{-fin}}) = (\mathrm{Ker}\ d_j)^{M\text{-fin}}$. □

Remark. This Lemma is unrelated to Casselman's conjectured comparison theorem, since they are under different conditions.

Corollary A.10 (Hochschild-Serre spectral sequence for nilpotent normal subgroups)**.** *The notation is as shown in Lemma A.9. Consider $V \in \mathcal{S}\mathrm{mod}_H$, if $\forall\ j \in \mathbb{Z}$, $\mathrm{H}_j^{\mathcal{S}}(N, V) \in \mathcal{S}\mathrm{mod}_L$, then we have*

$$\mathrm{H}_i(\mathfrak{l}_{\mathbb{C}}, M; \mathrm{H}_j(\mathfrak{n}_{\mathbb{C}}; V^{M\text{-fin}})) = \mathrm{H}_i^{\mathcal{S}}(L, \mathrm{H}_j^{\mathcal{S}}(N, V)),\ \forall\ i,\ j \in \mathbb{Z}.$$

Moreover, there exist convergent first quadrant spectral sequences:

$$E^2_{p,q} = \mathrm{H}_p^{\mathcal{S}}(L, \mathrm{H}_q^{\mathcal{S}}(N, V)) \implies \mathrm{H}_{p+q}^{\mathcal{S}}(H, V).$$

Proof. Following Lemma A.9 and Proposition 3.3,

$$\mathrm{H}_i(\mathfrak{l}_{\mathbb{C}}, M; \mathrm{H}_j(\mathfrak{n}_{\mathbb{C}}; V^{M\text{-fin}})) = \mathrm{H}_i(\mathfrak{l}_{\mathbb{C}}, M; \mathrm{H}_j^{\mathcal{S}}(N, V)^{M\text{-fin}}) = \mathrm{H}_i^{\mathcal{S}}(L, \mathrm{H}_j^{\mathcal{S}}(N, V)).$$

As to the second assertion, according to Theorem A.8, we have convergent first quadrant spectral sequences:

$$E^2_{p,q} = \mathrm{H}_p(\mathfrak{l}_{\mathbb{C}}, M; H_q(\mathfrak{n}_{\mathbb{C}}; V^{M\text{-fin}})) \implies H_{p+q}(\mathfrak{h}_{\mathbb{C}}, M; V^{M\text{-fin}})$$

Then the corollary follows from the first assertion and Proposition 3.3. □

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