# HOMEWORK 3

Students enrolled in this course are required to submit their solutions for this homework before 11:59 am on 18 March 2026, using one of the following methods.

(1) Send me an email with the solution attached.
(2) Submit a hard copy to me in person before class.

Please write your solutions clearly and neatly, and include sufficient explanations so that your reasoning is easy to follow.

**Problem 1.** Let $G$ be a Lie group with Lie algebra $\mathfrak{g}$, and let $a : G \times M \to M$ be a smooth action of $G$ on a manifold $M$. Denote by $\mathfrak{X}(M)$ the Lie algebra of smooth vector fields on $M$.

For $z \in \mathfrak{g}$, define a smooth vector field $a_*(z)$ on $M$ by

$$(a_*(z)f)(x) = \frac{d}{dt}\bigg|_{t=0} f\big(\exp(-tz) \cdot x\big),$$

where $t \in \mathbb{R}$, $x \in M$, and $f \in C^\infty(U)$ for some open set $U \subset M$ containing $x$.

Show that $a_* : \mathfrak{g} \to \mathfrak{X}(M)$ is a Lie algebra homomorphism by completing the following steps.

(1) Show that for each $z \in \mathfrak{g}$, the operator $a_*(z)$ defines a smooth vector field on $M$.
(2) Prove that the map
$$a_* : \mathfrak{g} \longrightarrow \mathfrak{X}(M)$$
is linear.
(3) Prove that
$$a_*([z, w]) = [a_*(z), a_*(w)] \qquad \text{for all } z, w \in \mathfrak{g}.$$

**Problem 2.** Let

$$J = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R}).$$

Prove that $J$ is not in the image of the exponential map

$$\exp : \mathfrak{sl}_2(\mathbb{R}) \to \mathrm{SL}_2(\mathbb{R}),$$

i.e. show that there is no $X \in \mathfrak{sl}_2(\mathbb{R})$ such that $e^X = J$. (Hint: if $e^X = J$, what are the eigenvalue of $X$?).

**Problem 3.** For a representation $V$ of a Lie algebra $\mathfrak{g}$, define the space of coinvariants by

$$V_{\mathfrak{g}} = V/\mathfrak{g}V,$$

where $\mathfrak{g}V$ is the subspace spanned by $xv$ for $x \in \mathfrak{g}$, $v \in V$.

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*Date*: August 24, 2026.

(1) Show that if $V$ is completely reducible, then the composition

$$V^{\mathfrak{g}} \hookrightarrow V \twoheadrightarrow V_{\mathfrak{g}}$$

is an isomorphism.

(2) Show that in general this is not so. (Hint: take $\mathfrak{g} = \mathbb{R}$ and an appropriate representation $V$.)

**Problem 4.** Let $G$ be a Lie group and $(\pi, V)$ a complex finite-dimensional irreducible representation of $G$. Suppose $(\cdot,\cdot)_1$ and $(\cdot,\cdot)_2$ are two $G$-invariant positive definite Hermitian forms on $V$.

Prove that there exists a positive real number $\lambda > 0$ such that

$$(v, w)_2 = \lambda(v, w)_1 \qquad \text{for all } v, w \in V.$$

In other words, show that the unitary structure on an irreducible unitary representation is unique up to a positive scalar multiple. (Hint: use Schur's Lemma)

**Problem 5.** Let $V$ be a finite-dimensional complex vector space. In the statement below, write $\mathrm{GL}(V)$ for the general linear group of $V$, which has a natural action on tensor powers of $V$.

**Definitions.**

- The *tensor power* $V^{\otimes m}$ is the $m$-fold tensor product of $V$ with itself, with the natural action of the symmetric group $S_m$ by permuting tensor factors.
- The *symmetric power* $S^m V$ may be defined either as the subspace of symmetric tensors

$$S^m V \;=\; \{\, T \in V^{\otimes m} :\ \sigma \cdot T = T \text{ for all } \sigma \in S_m \,\},$$

  or equivalently as the quotient $S^m V \cong V^{\otimes m}/I_{\mathrm{sym}}$ where $I_{\mathrm{sym}}$ is the subspace generated by all tensors of the form $v_1 \otimes \cdots \otimes v_m - v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(m)}$.
- The *exterior power* $\wedge^k V$ (also called the $k$-th alternating power) may be defined as the subspace of alternating tensors

$$\wedge^k V \;=\; \{\, T \in V^{\otimes k} :\ \sigma \cdot T = \mathrm{sgn}(\sigma) T \text{ for all } \sigma \in S_k \,\},$$

  or equivalently as the quotient $\wedge^k V \cong V^{\otimes k}/I_{\mathrm{alt}}$ where $I_{\mathrm{alt}}$ is generated by all tensors having two equal factors.

**Problem.** Show that

(1) $V$ is an irreducible representation of $\mathrm{GL}(V)$.
(2) $S^2 V$ is an irreducible $\mathrm{GL}(V)$-representation.
(3) $\wedge^2 V$ is an irreducible $\mathrm{GL}(V)$-representation.
(4) Show that $V \otimes V$ is completely reducible:

$$V \otimes V \;\cong\; S^2 V \oplus \wedge^2 V.$$

In fact, this problem can be viewed as a basic example of *Schur–Weyl duality.* If you are interested in learning more about its modern generalizations, you are warmly invited to attend the *Frontiers of Mathematics Lecture* by Professor Weiqiang Wang this Friday (March 06, 2026), 2:00–3:00 pm, at P4, Chong Yuet Ming Physics Building, HKU. For more details, please see: `https://hkumath.hku.hk/MathWWW/event/2026/Weiqiang_Wang_06032026.pdf`

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