HOMEWORK 2

Students enrolled in this course are required to submit their solutions for this homework before 11:59 am on 4 March 2026, using one of the following methods.

(1) Send me an email with the solution attached.
(2) Submit a hard copy to me in person before class.

Please write your solutions clearly and neatly, and include sufficient explanations so that your reasoning is easy to follow.

Problem 1. Let H be a closed Lie subgroup of the Lie group G. Show that if both H and G/H are connected, then G is connected.

Problem 2. Let $p : E \to X$ be an n-dimensional real vector bundle. Prove that E is trivial (i.e. $E \cong X \times \mathbb{R}^n$ as vector bundles) if and only if it admits n sections $s_1, \ldots, s_n : X \to E$ such that for every $x \in X$, the vectors

$$s_1(x), \cdots, s_n(x)$$

form a basis of $p^{-1}(x)$.

Problem 3. Let $\langle \cdot, \cdot \rangle$ denote the standard inner product on $\mathbb{R}^3$, defined by

$$\langle x, y \rangle \;=\; {}^t x \cdot y \quad \text{for } x, y \in \mathbb{R}^3,$$

where we view x, y as column vectors in $\mathbb{R}^3$, and ${}^t x$ denotes the transpose of x. It induces the Euclidean norm

$$\|x\| = \sqrt{\langle x, x \rangle}.$$

Let

$$\mathrm{SO}(3) := \big\{\, A \in M_3(\mathbb{R}) \mid \langle Ax, Ay \rangle = \langle x, y \rangle \text{ for all } x, y \in \mathbb{R}^3,\ \det A = 1 \,\big\},$$

be the Lie group of orientation-preserving linear isometries of $\mathbb{R}^3$. It acts on the unit sphere

$$S^2 := \{ v \in \mathbb{R}^3 \mid \|v\| = 1 \}$$

by

$$A \cdot v := Av.$$

(1) **(Transitivity)** Prove that this action is transitive.
(2) **(Stabilizer)** Compute the stabilizer of the point $e_3 = (0, 0, 1) \in S^2$. Show that it is isomophic to $\mathrm{SO}(2)$.
(3) **(Homogeneous space)** Conclude that $S^2 \cong \mathrm{SO}(3)/\mathrm{SO}(2)$ as smooth manifolds.

Date: August 24, 2026.

Problem 4. Let $f : G \to H$ be a smooth map between manifolds. Let X_1, X_2 be smooth vector fields on M, and Y_1, Y_2 smooth vector fields on N. Assume that

$$df \circ X_i = Y_i \circ f, \quad i = 1, 2,$$

that is, each X_i is f-related to Y_i. Prove that the Lie bracket is also f-related:

$$df \circ [X_1, X_2] = [Y_1, Y_2] \circ f.$$

Problem 5. Let V be a 2-dimensional real vector space with basis $\{x, y\}$. Define a bilinear bracket $[\cdot,\cdot] : V \times V \to V$ on the basis elements by

$$[x, x] = [y, y] = 0, \qquad [x, y] = y, \qquad [y, x] = -y,$$

and extend bilinearly to all of V.

(1) Show that $(V, [\cdot,\cdot])$ is a real Lie algebra.
(2) Show that any 2-dimensional non-abelian real Lie algebra is isomorphic to $(V, [\cdot,\cdot])$.

Email address: gengzb@hku.hk